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ON THE DIFFUSION OF TIDES INTO PERMEABLE ROCK OF FINITE DEPTH*

BY

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1. Introduction. It has been observed in the irrigation wells of the Hawaiian Islands that the water-level fluctuations have frequency components corresponding to those of the ocean tides [1]. This phenomenon was analyzed by Carrier and Munk [2], assuming the observed ground-water fluctuations to represent a diffusive transmission of the tidal disturbances through the porous volcanic structure of the island. The purpose of the investigation was to use the results in estimating the permeability of the porous medium.

In [2] it was assumed that the porous medium was infinitely deep. In actual fact, however, there will be an essentially impenetrable bounding surface (see Fig. 1). This paper is concerned with the analysis of the same problem treated in [2], but taking account of the bounding bottom surface. Numerical computations are carried out for several values of the dimensionless depth. Also the limiting case of shallow water theory is studied. Using the results of the infinite depth, shallow depth, and finite depth theory it is possible from the graphs given in Figs. 2 and 3 to estimate the amplitude and phase lag in the fluctuations of the ground-water as a function of the distance inland for various values of the dimensionless depth. It is found that for the values of the physical parameters which are probably of most concern the infinite depth theory gives satisfactory results in the region of interest.

2. Formulation of the problem. Although the formulation and the first part of the

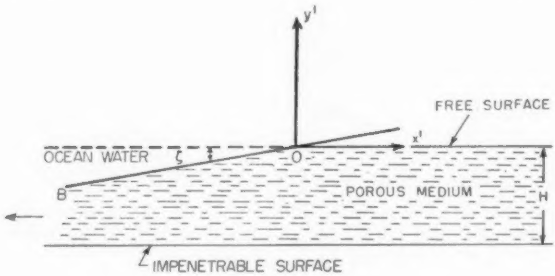

FIG. 1.

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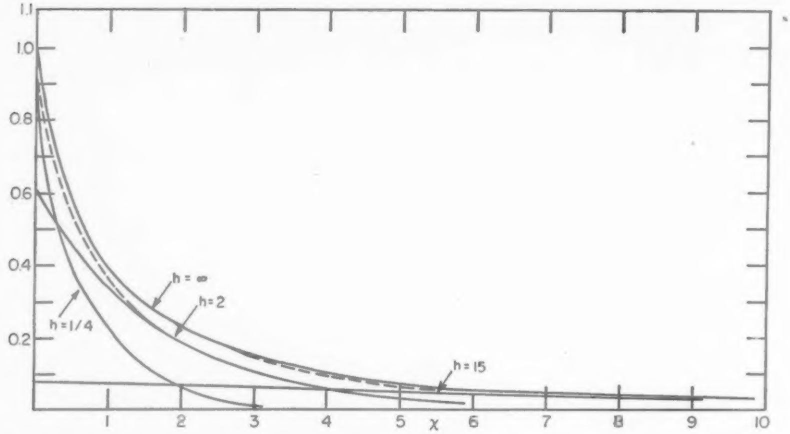

Fig. 2. Ground-water amplitude *vs.* distance inland, x. (Dotted lines indicate the expected correction to the computed curves when all modes are considered)

analysis of this problem follows quite closely that related in [2] it is convenient, for the sake of completeness, to repeat part of that work here. As the water in the ocean bounded by Oy' and OB (see Fig. 1) rises and falls about its mean level OA, the pressure on the line OB varies. Corresponding to this periodic change in pressure on OB we can expect periodic fluctuations in the free surface of the ground-water, i.e. Ox'.

The equations governing the motion of the fluid in the porous medium are the conservation of mass

$$\operatorname{div}(\rho \mathbf{v}) = -\frac{\partial}{\partial t}(\rho\theta), \tag{2.1}$$

and Darcy's law which replaces the conservation of momentum law (see [3]),

$$\mathbf{v} = -(k/\mu)\operatorname{grad}(p - p_0). \tag{2.2}$$

Here ρ and μ are the density and the viscosity of the fluid; θ and k are the porosity and the permeability of the material; $\mathbf{v}$ is the velocity with components u and v in the x' and y' direction; p is the gauge pressure; $p_0 = -\rho_0 g y'$ is the pressure when the fluid motion is zero; ρ_0 is the mean density; and subscript notation indicates partial differentiation. The simple compressibility law used by Carrier and Munk is

$$\rho\theta = \rho_0\theta_0\{1 + \delta(p - p_0)\}, \tag{2.3}$$

where δ is essentially $(\rho_0 c^2)^{-1}$ with c the speed of sound in the fluid. It should be pointed out that Eq. (2.2) says that the pressure gradient is proportional to a velocity rather than an acceleration as in the Navier-Stokes equation. As a result we will obtain finally an equation of the diffusion type rather than a wave equation; hence the free surface amplitude will decay in x'.

The boundary conditions expressed in terms of the pressure p are

$$p = 0, \quad \text{on the free surface,} \tag{2.4a}$$

$$p = -\rho_0 g y'_{OB} + q_1 e^{i\omega t} \text{ on } OB, \tag{2.4b}$$

$$\partial(p - p_0)/\partial y' = 0 \text{ on } y' = -H, \tag{2.4c}$$

$p = 0$, on the free surface,
where H is the depth of the porous medium. The last boundary condition results from the requirement that the normal component of velocity be zero on the impenetrable bottom. The pressure q_1 is of course directly proportional to the tidal-wave amplitude.

If we let $q = p - p_0$, and combine Eqs. (2.1), (2.2) and (2.3) we obtain

$$\Delta q = (\mu\theta_0\delta/k)q_t\ , \tag{2.5}$$

where Δ is the Laplacian operator. If we denote by $\eta(x', t)$ the y' coordinate of the free surface, Eq. (2.4a) implies $q(x', \eta, t) = \rho_0 g\eta$, but on the free surface $\eta_t = v/\epsilon$, hence using Eq. (2.2) our boundary condition (2.4a) may be expressed as

$$q_t + (\rho_0 gk/\mu\theta_0)q_{y'} = 0 \quad \text{on} \quad y' = \eta, x > 0. \tag{2.6a}$$

Actually as in the usual linear theory of water waves this boundary condition is to be applied on $y = 0$. The boundary conditions (2.4b) and (2.4c) may be written as

$$q(\text{on } OB) = q_1 e^{i\omega t}, \tag{2.6b}$$

$$q_{y'} = 0, y' = -H. \tag{2.6c}$$

We shall only solve this problem in the case that the line OB occupies the half-line $y = 0, x \leq 0$. That is we take $\zeta = 0°$ (see Fig. 1). Actually this is fairly realistic since ζ is probably of the order of 5° or so.

Finally if we introduce the following dimensionless variables

$$\left.\begin{aligned} &\tau = \omega t, \qquad x = x'/L, \qquad y = y'/L, \qquad h = H/L, \\ &L = (\rho_0 gk)/(\mu\theta_0\omega), \qquad \epsilon = (\rho_0^2 g^2 k\delta)/(\mu\theta_0\omega), \qquad q = q_1\varphi(x, y)e^{i\omega t}, \end{aligned}\right\} \tag{2.7}$$

we obtain

$$\Delta\varphi - i\epsilon\varphi = 0, \tag{2.8}$$

with the boundary conditions

$$\varphi_y + i\varphi = 0, \qquad y = 0, \qquad x > 0, \tag{2.9a}$$

$$\varphi = 1, \qquad y = 0, \qquad x < 0, \tag{2.9b}$$

$$\varphi_y = 0, \qquad y = -h, \qquad -\infty < x < \infty. \tag{2.9c}$$

The free surface $\eta(x, t)$ is $(\rho_0 g)^{-1}\, q(x, 0, t)$, but from Eqs. (2.7) $q = q_1\varphi(x, y) \exp(i\omega t)$ so

$$\eta(x, t) = \frac{q_1}{\rho_0 g}\varphi(x, 0)e^{i\omega t}\ x > 0. \tag{2.10}$$

So the problem of determining the free surface is exactly that of determining $\varphi(x, 0)$. The combination of parameters $q_1/\rho_0 g$ is the maximum height of the tidal-wave measured from $y' = 0$.

Before proceeding to a solution of the problem defined by Eqs. (2.8) and (2.9) it is perhaps worthwhile to mention briefly the size of the parameters which appear in this problem. We have $\mu/\rho_0 = 0\ (10^{-2}cm^2/\text{sec})$, $k = 0(5 \times 10^{-6}\text{cm}^2)$, $\theta = 0(.20)$, $g = 980$ cm/sec^2, and ω for a twenty-four hour tide is $2\pi/24$ hours, hence $L = 0(1000 \text{ ft})$. Since

c is 0(5000 ft/sec) for water, ϵ is a very small number, $0(10^{-4})$. Finally, a reasonable value for the depth of the ocean is about three miles so h may be as large as 15.

3. Shallow water theory. Before considering the general problem given by Eqs. (2.8) and (2.9) let us look at the limiting case in which the depth H is small enough that we can neglect variations in the y' direction, and also set $v \equiv 0$. Then $u(x', t)$ represents an averaged velocity across the section $-H < y' < 0$. If we assume incompressibility, i.e. $\delta = 0$, the conservation of mass equation appropriate to this situation is

$$H\rho_0 u_{x'} = -\rho_0\theta_0\eta_t \,. \tag{3.1}$$

Darcy's equation reduces to

$$u = -(k/\mu)q_{x'} \,. \tag{3.2}$$

Since there is no variation in the y direction our condition that $q = \rho_0 g\eta$ on the free surface must hold throughout the strip $-H < y' < 0$, $x > 0$. Using this and Eqs. (3.1) and (3.2) we obtain

$$q_{x'x'} - (\mu\theta_0/k\rho_0 gH)q_t = 0, \qquad x' > 0. \tag{3.3}$$

The condition that $q = q_1 \exp(i\omega t)$ for $x \leq 0$ is now applied at $x = 0$; consequently we set $q = q_1\varphi(x') \exp(i\omega t)$. Then Eq. (3.3) becomes

$$\varphi_{x'x'} - i(HL)^{-1}\varphi = 0, \qquad x' > 0. \tag{3.4}$$

An appropriate solution of Eq. (3.4) satisfying a finiteness condition at infinity is

$$\varphi(x', t) = \exp[-(i/HL)^{\frac{1}{2}}x'].$$

Hence

$$\eta(x', t) = (q_1/\rho_0 g) \exp\{-x'/(2HL)^{\frac{1}{2}} + i[\omega t - x'/(2HL)^{\frac{1}{2}}]\}, \tag{3.5a}$$

$$= (q_1/\rho_0 g) \exp\{-x/(2h)^{\frac{1}{2}} + i[\omega t - x/(2h)^{\frac{1}{2}}]\}. \tag{3.5b}$$

Actually, in order for this theory to be valid not only must the wave length of the disturbance be large compared to H as in the usual shallow water theory but also H must be small compared to the other natural length scale, L, which appears in the problem, i.e. h must be small. This can be seen by an examination of the behaviour of the solution of the general problem. This is done in Sec. 5 where it is found that for $h \leq 1/4$ we can expect the shallow water theory to be quite accurate. The amplitude and phase lag of $\rho_0 g\eta(x, t)/q_1$ are plotted in Figs. 2 and 3 as a function of x for $h = 1/4$.

4. Solution of the problem. To solve the problem defined by Eqs. (2.8) and (2.9) we shall use the method of Fourier transforms and the Wiener-Hopf technique. Let

$$\Phi(\xi, y) = \int_{-\infty}^{\infty} e^{-i\xi x}\varphi(x, y)\, dx. \tag{4.1}$$

Then the transform of Eq. (2.8) is

$$\Phi_{yy} - (\xi^2 + i\epsilon)\Phi = 0. \tag{4.2}$$

A solution of this equation satisfying the boundary condition (2.9c) is

$$\Phi(\xi, y) = A(\xi) \cosh\{(y + h)C\}, \tag{4.3}$$

where $C = (\xi^2 + i\epsilon)^{1/2}$ and $A(\xi)$ is to be determined by satisfying the remaining boundary conditions.

Let

$$g_1(x) = \begin{cases} \lim_{a\to 0} e^{ax} & x < 0 \\ 0 & x > 0 \end{cases}, \tag{4.4a}$$

$$g_2(x) = \begin{cases} 0 & x < 0 \\ \varphi(x, 0) & x > 0 \end{cases}, \tag{4.4b}$$

and

$$f(x) = \varphi_y(x, 0) + i\varphi(x, 0). \tag{4.5}$$

It is clear that $\varphi(x, 0) = g_1(x) + g_2(x)$; hence*

$$\left.\begin{aligned} \Phi(\xi, 0) = A(\xi) \cosh Ch &= G_1(\xi) + G_2(\xi), \\ &= (a - i\xi)^{-1} + G_2(\xi). \end{aligned}\right\} \tag{4.6}$$

Also using Eq. (4.3)

$$F(\xi) = (C \sinh Ch + i \cosh Ch)A(\xi). \tag{4.7}$$

Combining Eqs. (4.6) and (4.7) we obtain

$$F(\xi) = K(\xi)\{G_1(\xi) + G_2(\xi)\}, \tag{4.8}$$

where

$$K(\xi) = \frac{C \sinh Ch + i \cosh Ch}{\cosh Ch}. \tag{4.9}$$

If we recall from Sec. 2 that we wish to determine $\varphi(x, 0)$ it is clear from Eq. (4.4) that our problem is now that of determining $g_2(x)$ and hence $G_2(\xi)$. To determine $G_2(\xi)$ using Eq. (4.8) we shall use the Wiener-Hopf technique. This technique has been used to treat similar problems (see for instance [2, 4, 5]); consequently the analysis will only be briefly outlined here. First $G_1(\xi)$ is analytic in the upper half plane, (*UHP*), $Im(\xi) > -a$; $G_2(\xi)$ is analytic in the *LHP* and $F(\xi)$ is analytic in the *UHP*. The function $K(\xi)$ is analytic and non-vanishing in a strip containing the real axis. This will be seen clearly at the end of this section where $K(\xi)$ is represented as the quotient of two infinite products. It might be noted that though $C = (\xi^2 + i\epsilon)^{1/2}$ is a multivalued function, $K(\xi)$ as defined by Eq. (4.9) consists only of even terms and hence does not have any branch points. Assuming for the moment that we can write $K(\xi)$ as $K_-(\xi)/K_+(\xi)$ where $K_-(\xi)$ is analytic and non-vanishing in the *LHP*, and $K_+(\xi)$ is analytic and non-vanishing in the *UHP* we can rewrite Eq. (4.8) as

$$F(\xi)K_+(\xi) - K_-(-ia)G_1(\xi) = \{K_-(\xi) - K_-(-ia)\}G_1(\xi) + K_-(\xi)G_2(\xi). \tag{4.10}$$

The left hand side of this equation is analytic in the *UHP*, the right hand side is analytic in the *LHP* and they agree in a common strip of analyticity. Hence Eq. (4.10) defines an entire function $E(\xi)$. We shall show shortly that $K_-(\xi) = 0(\xi^{1/2})$ as $|\xi| \to \infty$, $Im(\xi) < 0$ and $K_+(\xi) = 0(\xi^{-1/2})$ as $|\xi| \to \infty$, $Im(\xi) > 0$. Using this and investigating order conditions at infinity we can show that $E(\xi) = 0$, consequently

$$G_2(\xi) = \left\{\frac{K_-(-ia)}{K_-(\xi)} - 1\right\} G_1(\xi). \tag{4.11}$$

*Capital letters are used to denote the Fourier transform.

It is now necessary to determine $K_-(\xi)$ and $K_+(\xi)$. The splitting of $K(\xi)$ is done in a manner exactly analogous to that used by Heins in [4] and [5]. Using the infinite product representation of cosh z, (see [6]) we have

$$M(\xi) = \cosh Ch = \prod_{n=0}^{\infty} \{1 + (2Ch)^2/(2n + 1)^2\pi^2\} = m(\xi)m(-\xi), \tag{4.12}$$

where

$$\begin{aligned} m(\xi) = \prod_{n=0}^{\infty} \{[1 + (4i\epsilon h^2)/(2n + 1)^2\pi^2]^{\frac{1}{2}} \\ + i2h\xi/(2n + 1)\pi\} \exp [- i2h\xi/(2n + 1)\pi]. \end{aligned} \tag{4.13}$$

We have inserted the exponentials to insure absolute convergence of the infinite products defining $m(\xi)$ and $m(-\xi)$ in the UHP and LHP respectively. If we write $M(\xi)$ as $M_-(\xi)/M_+(\xi)$ it is clear that $M_-(\xi) = m(\xi)$ has no zeros or poles in the LHP. Similarly $1/M_+(\xi) = m(-\xi)$ has no zeros or poles in the UHP.

The function $L(\xi) = C \sinh Ch + i \cosh Ch$ has zeros at $Ch = \pm i\beta_n$, $n = 0, 1, 2, \cdots$, where the β_n are complex numbers lying in the first quadrant. For n large they may be determined by the asymptotic relation $\beta_n = n\pi + ih/n\pi + 0\cdot([n\pi]^2)$. We may write $L(\xi)$ as

$$L(\xi) = i \prod_{n=0}^{\infty} \{1 + (Ch)^2/\beta_n^2\} = il(\xi)l(-\xi), \tag{4.14}$$

where

$$\begin{aligned} l(\xi) = \{[1 + (i\epsilon h^2)/\beta_0^2]^{\frac{1}{2}} + i\xi h/\beta_0\} \prod_{n=1}^{\infty} \{[1 + (i\epsilon h^2)/\beta_n^2]^{\frac{1}{2}} + i\xi h/\beta_n\} \\ \exp (- i\xi h/n\pi). \end{aligned} \tag{4.15}$$

Again we have inserted the exponentials in order to insure absolute convergence in the appropriate half planes. If we write $L(\xi)$ as $L_-(\xi)/L_+(\xi)$ and take $L_-(\xi) = l(\xi)$, $1/L_+(\xi) = il(-\xi)$ it is clear that $L_-(\xi)$ is free of zeros and poles in the LHP and $L_+(\xi)$ is free of zeros and poles in the UHP.

Consequently we have

$$K_-(\xi) = \exp \{\chi(\xi)\}L_-(\xi)/M_-(\xi) = \exp \{\chi(\xi)\} l(\xi)/m(\xi), \tag{4.16a}$$

$$K_+(\xi) = \exp \{\chi(\xi)\}L_+(\xi)/M_+(\xi) = \exp \{\chi(\xi)\} m(-\xi)/il(-\xi). \tag{4.16b}$$

We shall choose the factor $\exp \{\chi(\xi)\}$ introduced in Eqs. (4.16a) and (4.16b) in such a manner that $K_-(\xi)$ and $K_+(\xi)$ have algebraic behaviour as $|\xi| \to \infty$ in the LHP and UHP respectively.

To investigate the behaviour of $K_-(\xi)$ for $Im(\xi) < 0$, $|\xi| \to \infty$ we first note that the terms involving ϵ may be neglected against unity for $|\xi| \to \infty$. Since $\beta_n \to n\pi$ as $n \to \infty$ we have that $K_-(\xi)$ is of the order

$$\exp \{\chi(\xi)\}(1 + i\xi h/\beta_0) \prod_{n=1}^{\infty} \{(1 + w/n) \exp (- w/n)\} \Big/$$

$$\prod_{n=1}^{\infty} \{[1 + 2w/(2n + 1)] \exp [- 2w/(2n + 1)]\},$$

where $w = i\xi h/\pi$. Now using the relation that

$$1/\Gamma(w) = we^{\gamma w} \prod_{n=1}^{\infty} (1 + w/n)e^{-w/n},$$

and Stirling's asymptotic formula for the gamma function, (see [6]) we obtain

$$K_-(\xi) = 0\{w^{\frac{1}{2}} \exp [\chi(\xi) + w ln 4]\},$$

for $Im(\xi) < 0$, $|\xi| \to \infty$. So choosing $\chi(\xi) = -w\, ln\, 4 = -(i\xi h\, ln\, 4)/\pi$ we have that $K_-(\xi) = 0(\xi^{1/2})$ as $|\xi| \to \infty$, $Im(\xi) < 0$. A similar argument will show that $K_+(\xi) = 0(w^{-1/2})$ for $|\xi| \to \infty$, $Im(\xi) > 0$. With these order relations it is not difficult to show that $E(\xi)$ is zero as mentioned earlier; and hence we obtain Eq. (4.11) for $G_2(\xi)$ where $K_-(\xi)$ is defined by Eq. (4.16a). In particular it can be seen from Eq. (4.16a) and the definitions of $1(\xi)$ and $m(\xi)$ that $K_-(0) = 1$.

Using the usual inversion formula we have that

$$G_2(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i\xi x}\{[K_-(-ia) - K_-(\xi)]/(a - i\xi)K_-(\xi)\}\, d\xi, \tag{4.17a}$$

$$= (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i\xi x} \frac{K_-(-ia) \cosh Ch - K_+(\xi)K(\xi)}{(a - i\xi)K_+(\xi)K(\xi)}\, d\xi. \tag{4.17b}$$

In the limit as $a \to 0$ it is clear from Eq. (4.17a), that $G_2(\xi)$ is not singular at the origin; hence we may actually take the real axis as our path of integration in evaluating $g_2(x)$. Of course we shall actually close our path of integration in the *UHP* when $x > 0$ and in the *LHP* when $x < 0$. For the case $x > 0$ it is convenient to use Eq. (4.17b) for evaluating $g_2(x)$. Since $K_+(\xi)$ is non-vanishing in the *UHP*, $g_2(x)$ will be simply the sum of the residues at the poles of the integrand which occur at $Ch = i\beta_n$, $n = 0, 1, \cdots$. Carrying out this straightforward computation we obtain in the limit

$$g_2(x) = h^{-1} \sum_{n=0}^{\infty} \left(\frac{\beta_n}{\alpha_n}\right)^2 \frac{e^{-\alpha_n x}}{(\beta_n^2 + ih - h^2)K_+(i\alpha_n)}, \tag{4.18}$$

where $\alpha_n = (\beta_n^2/h^2 + i\epsilon)^{1/2}$. In the particular case that the fluid is incompressible, i.e. $\epsilon = 0$, $\beta_n/\alpha_n = h$ and Eq. (4.18) becomes

$$g_2(x) = h \sum_{n=0}^{\infty} \frac{e^{-\alpha_n x}}{(h^2\alpha_n^2 + ih - h^2)K_+(i\alpha_n)}, \qquad \alpha_n = \beta_n/h. \tag{4.19}$$

5. Numerical computations and discussion. In this section we shall only be concerned with the case in which the fluid may be considered as incompressible, then $g_2(x)$ is given by Eq. (4.19). In [2] a few values of $g_2(x)$ were computed for $\epsilon = .01$ and compared to the $\epsilon = 0$ case; the amplitude and phase lag in the ground-water fluctuation for $\epsilon = .01$ were slightly lower than for $\epsilon = 0$. Since ϵ is however $0(10^{-4})$ we should expect very little error in actually setting $\epsilon = 0$.

First let us determine when the shallow water theory solution given by Eqs. (3.5) may be expected to be valid. In order for $g_2(x)$ as given by Eq. (4.19) to agree with the shallow water solution it is necessary that $\alpha_0 \sim (i/h)^{1/2}$ as $h \to 0$, and also that all the coefficients of the higher order terms[1] must approach zero. It can be shown with little

[1]We shall refer to the term $\exp(-\alpha_0 x)$ in Eq. (4.19), which is the dominating term as $x \to \infty$, as the fundamental term.

difficulty from an investigation of the transcendental equation $C \sinh Ch + i \cosh Ch = 0$ that for small h, $\beta_0 \sim (ih)^{1/2}$ and hence $\alpha_0 \sim (i/h)^{1/2}$. Also upon noting that for small h, $\beta_n \sim n\pi$ for $n \geq 1$, it can be seen from Eq. (4.19) with the aid of the representation of $K_+(i\alpha_n)$ given in Eq. (5.1) that all the coefficients of the higher order terms do approach zero as $h \to 0$. Hence $g_2(x)$ as given by Eq. (4.19) does approach the shallow water solution as $h \to 0$. To determine quantitatively when Eq. (3.5b) is valid we have computed β_0 and α_0 as a function of h. Also the ratios of the wave length predicted by the shallow water theory, $\lambda = 2\pi/(2HL)^{1/2}$, to H and that of the fundamental mode, $\lambda_0 = 2\pi/Im(\alpha_0)$, to H have been computed. These results are given in Table 1 and

TABLE 1.

h is the dimensionless depth, $\lambda = 2\pi(2HL)^{1/2}$ is the wave length for shallow water theory, $\lambda_0 = 2\pi L/Im(\alpha_0)$ is the wave length of the fundamental mode for finite bottom theory.

h	β_0	$\alpha_0(\epsilon = 0)$	λ/H	λ_0/H
$\to 0$	$\to (ih)^{1/2}$			
.25	.3676 + i .3382	1.4704 + i 1.3528	17.76	18.57
.50	.5376 + i .4548	1.0752 + i .9096	12.57	13.81
1.00	.8004 + i .5702	.8004 + i .5702	8.88	11.09
2.00	1.1828 + i .5832	.5914 + i .2916	6.28	10.77
3.00	1.3739 + i .4775	.4579 + i .1592	4.85	13.16
5.00	1.5033 + i .3090	.3007 + i .0618	3.97	20.03
10.00	1.5547 + i .1569	.1555 + i .0157	2.81	40.04
15.00	1.5638 + i .1046	.1043 + i .0070	2.29	59.83

graphically in Fig. 4. It appears from Fig. 4 that we may expect the shallow water theory to be accurate over the entire range of x for $h \leq 1/4$.

In order to compute $g_2(x)$ for various values of h it is necessary to cast $K_+(i\alpha_n)$ into a form more suitable for numerical analysis than that given by Eq. (4.16a). This can be done in a straightforward manner by using the infinite product representation of the gamma function. We obtain, when $\epsilon = 0$, that

$$K_+(i\alpha_n) = \frac{\beta_n[\Gamma(\beta_n/\pi)]^2 \exp\,[(\beta_n \ln 4)/\pi]}{2\pi i(1 + \beta_n/\beta_0)\Gamma(2\beta_n/\pi)} \left\{ \prod_{m=1}^{\infty} [(1 + \beta_n/\beta_m)/(1 + \beta_n/m\pi)] \right\}^{-1}. \quad (5.1)$$

In any actual numerical computation the infinite product in Eq. (5.1) is, of course, to be replaced by a finite number of terms (recall that $\beta_m \to m\pi$ as $m \to \infty$). The number of terms that is required to give an accurate answer is of course dependent upon h and β_n.

TABLE 2.

n	$\beta_n(h = 2)$	$\beta_n(h = 15)$
0	1.1828 + i .5832	1.5638 + i .1046
1	3.3106 + i .6499	4.6886 + i .3234
2	6.3014 + i .3277	7.8016 + .5749
3	9.4248 + i .2138	10.8705 + i .9070
4		13.7139 + i 1.3629
5		16.1332 + i 1.4236
6		18.9644 + i 1.0572

In this paper $g_2(x)$ was computed for $h = 2$ and $h = 15$. Values of β_n for $h = 2$ and $h = 15$ are given in Table 2. Let us first consider the case $h = 2$. Then the real parts of α_0 and α_1 are given by .59 and 1.65 respectively hence we may expect the fundamental term to give an accurate result for $x \geq 2$. Carrying out the necessary computations we obtain

$$\begin{aligned} \frac{\rho_0 g}{q_1}\,\eta(x, t) &= .61 \exp(-\ .59x) \exp[i(\omega t - .885 - .292x)] \\ &\quad + 0\,[\exp(-\ \alpha_1 x)]; \qquad h = 2. \end{aligned} \tag{5.2}$$

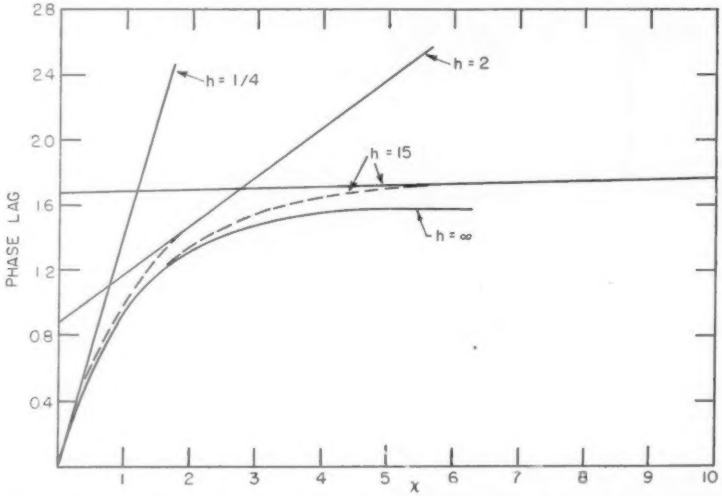

FIG. 3. Ground-water phase lag *vs.* distance inland, x. (Dotted lines indicate the expected correction to the computed curves when all modes are considered)

In Figs. 2 and 3 the amplitude and phase lag in the ground-water fluctuation have been plotted. The extrapolation of the results to $x = 0$ are indicated by dotted lines. Actually it is not difficult to obtain another term in the series, but unless particular quantitative information is desired for this value of h it hardly seems necessary to do that. It might be mentioned that three terms were more than sufficient in evaluating the infinite product in Eq. (5.1).

In the case that $h = 15$, the fundamental term can only be expected to be accurate for $x \geq 7$. We obtain

$$\begin{aligned} \frac{\rho_0 g}{q_1}\,\eta(x, t) &= .074 \exp(-\ .104x) \exp[i(\omega t - 1.68 - .007x)] \\ &\quad + 0\,[\exp(-\ \alpha_1 x)]; \qquad h = 15. \end{aligned} \tag{5.3}$$

In order to obtain results valid for $x = 1$ or 2 when $h = 15$ would probably require the computation of three or four terms of the series. However in view of the results for $h = 2$ and these results for $x \geq 7$ it is clear that the amplitude curve for $h = 15$ will lie almost exactly on the curve given by the infinite depth theory[2] (see Fig. 2).

[2]The amplitude and phase lag curves for $h = \infty$ have been taken from the results given in [2].

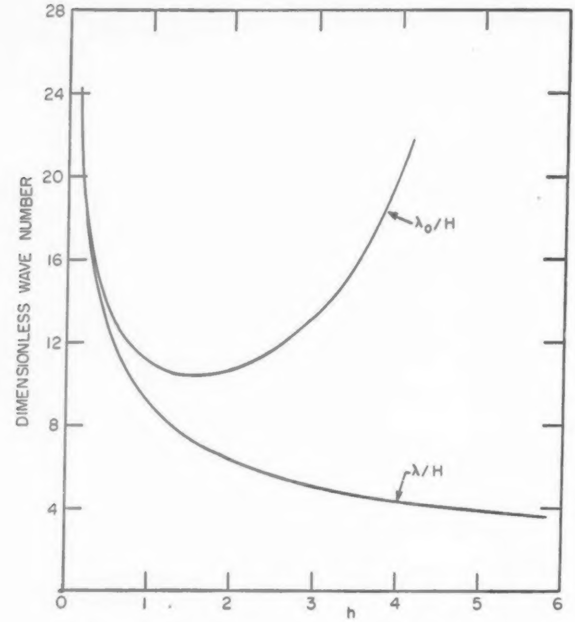

FIG. 4. Dimensionless wave number *vs.* h; λ is the wave length predicted by shallow water theory; λ_0 is the wave length of the fundamental mode predicted by finite bottom theory.

It is interesting to note that the shallow water theory and finite depth theory predict an exponential decay in x for the amplitude of the ground-water fluctuation; this is in contrast to the algebraic decay, (like x^{-1}), predicted by the infinite depth theory. Also the phase lag predicted by the shallow water theory and finite depth theory continues to increase with x, while that predicted by the infinite depth theory approaches $\pi/2$ with increasing x. (This is clearly illustrated in Fig. 3). That $g_2(x)$ as given by Eq. (4.19) approaches the infinite depth result given in [2] as $h \to \infty$, cannot be seen easily from (4.19). However an examination of $K(\xi)$ as given by Eq. (4.9) shows that as $Ch \to \infty$, $K(\xi) \to i + (\xi^2 + i\epsilon)^{1/2}$ which we might denote by $K_\infty(\xi)$. This function is the one that occurs in [2]. It is interesting to note that $K_\infty(\xi)$ has singularities of the branch point type, and in the limit as $\epsilon \to 0$ these singularities will occur at the origin. This explains the algebraic behaviour of $\eta(x, t)$ for $h = \infty$. In contrast for any finite value of h the strip of analyticity of $K(\xi)$ is finite even when $\epsilon \to 0$, and its singularities are poles rather than branch points; hence the exponential sort of behaviour for $\eta(x, t)$ for finite h.

A plausible physical explanation for the fact that the amplitude curves for the ground-water fluctuation lie continuously below one another as h decreases (see Fig. 2) is the following. Imagine that our porous medium and fluid occupy the strip $-H < y < 0$, $-\infty < x < \infty$. Suppose that we apply a uniform pressure on the half line $y = 0$, $-\infty < x < 0$; then fluid in the left half strip will be forced through the gap $-h < y < 0$ and the free surface given originally by $y = 0$, $x \geq 0$ will rise. The amount of fluid that can be forced through this gap, and hence the effect that the pressure variation can have

on the free surface, is proportional to the gap distance, h. So with decreasing h the amplitude of the free surface fluctuation is lower and dies out more quickly.

6. Acknowledgment. The author would like to express his appreciation to Professor G. F. Carrier for suggesting this problem and for his continued interest during several stimulating discussions. He is also indebted to Professor R. E. Kronauer for his helpful comments.

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BOOK REVIEWS

Geometric algebra. By E. Artin. Interscience Publishers, Inc., New York and London, 1957. x + 214 pp. $6.00.

This incisive monograph leads one by rapid and brilliant steps through the author's own algebraic foundation of affine and projective geometry, into a bird's-eye view of orthogonal and "symplectic" geometries over general fields and, after a brief discussion of determinants over non-commutative fields, the structure of what modern algebraists now refer to as the "classical" groups associated with these geometries. The generality and penetration of the results obtained are truly remarkable. Thus, "fields" are not even assumed to be commutative; "metric structures" (generalized dot products) need be neither symmetric (though $\mathbf{x} \cdot \mathbf{y} = 0$ is assumed to imply $\mathbf{y} \cdot \mathbf{x} = 0$) nor positive. Another interesting feature is the systematic use of coordinate-free methods.

On the other hand, "applied" mathematicians primarily interested in the real and complex number systems will probably find the book unnecessarily abstract and sophisticated, and the author's antipathy to coordinates and matrices (p. 13) exaggerated. Therefore, it is perhaps well to recall that all mathematics is potentially applicable, and that the volume being reviewed is at least to be admired as an original *tour de force*, among the very best works of its kind.

GARRETT BIRKHOFF

On human communication. A review, a survey, and a criticism. By Colin Cherry. Published jointly by the Technology Press, M. I. T., Mass., John Wiley & Sons, Inc., New York, and Chapman and Hall, Ltd. London, 1957. xiv + 333 pp. $6.75.

This is a very attractive and useful book. On pages 2 and 3, the author explains his purpose, in a statement which we regretfully abbreviate, but using his own words: "At the time of writing, the various aspects of communication, as they are studied under the different disciplines, by no means form a unified study; there is a common ground which shows promise of fertility, nothing more./ We shall attempt a review, a survey and a criticism of the study as it is being developed./ This book is introductory./ The book is written for that curious person, the 'general reader'./ We are seeking to extract from/ different sciences—linguistics, phonetics, communication theory, semantics, psychology/ the common related concepts and ideas concerning communication."

The applied mathematician, being usually a man who likes to look over the fence, will derive both pleasure and profit from this very clearly written, conscientious, and unassuming survey. He will get a good idea of the work of a very interesting group of people, located at Harvard and M.I.T., experts in the various disciplines quoted above, who for the last fifteen years, in a remarkable cooperative effort, have built up this broad common field of communication. The present book is the first volume of a series, "Studies in Communication", published by M.I.T. Press and Wiley, and the editors of the series, William N. Locke, Leo L. Beranek and Roman Jakobson, are prominent members of the group. Claude Shannon, who has done so much in the mathematical part of the field, has recently come to Cambridge. Prominent visitors from England have been A. Tustin, D. Gabor and the author of this book. Their work and that of more than a hundred people, from Adrian to Zipf, is discussed in the book and referenced in a marvelous bibliography of 367 titles.

To the applied mathematician who, after reading *On Human Communication*, would like to go further into communication theory, we would recommend reading, in order of publication, the excellent books of Shannon and Weaver (Illinois Press, 1949) Woodward (Pergamon-McGraw-Hill, 1953) and Brillouin (Academic Press, 1956). Some knowledge of cryptography (a habit-forming pastime) is very helpful.

P. LE CORBEILLER

(*Continued on p. 394*)

ON THE PERIODIC SOLUTIONS OF THE FORCED OSCILLATOR EQUATION*

BY

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Introduction. The phenomenon of "subharmonic vibrations" has been considered by many authors. The research on the subject goes back nearly one hundred years, beginning (probably) with Melde [14]†, Helmholtz [5], and Lord Rayleigh [17], all before the turn of the century, and continuing uninterruptedly to the present. For a fairly complete bibliography on this subject, the reader is referred to the classical paper on non-linear engineering problems by von Karman [8], and the books by Stoker [20], Minorski [15] and McLachlan [13].

In this paper, we shall discuss a single degree of freedom system whose mechanical model might be a mass under the action of an "elastic force" (linear or non-linear, restoring and/or exciting) and of a simple harmonic forcing function of frequency ω. Moreover, we assume that small quantities of positive damping are present in the physical system, but absent from the equation of motion. Such damping can be shown, in general, to reduce the free vibrations to negligibly small amplitudes in a finite time. Consequently, the periodic solutions of this system are those usually referred to as "steady state vibrations." We are assuming here that the effect of small damping on the motion is slight, but later on we shall prove this to be the case.

It follows that the equation of motion of the system is the oscillator equation

$$x^{\cdot\cdot} + f(x) = P_0 \cos \omega t, \tag{1}$$

where P_0 and ω are non-vanishing, finite, real constants, x is a dependent real variable and t is the independent variable time. For the present, the function $f(x)$ remains undefined.

Equation (1) and its harmonic and subharmonic solutions have been explored widely and by a considerable variety of methods, but it seems that the treatment was restricted in most cases to almost linear, and odd, $f(x)$. Ludeke [12] is one of the early investigators who has considered some cases, and experimented with some models, in which the departure of the elastic force from linearity is significant. However, he only considers very special cases; for instance, his treatment of the 1/3 order subharmonic vibration is restricted to special values of the forcing frequency. Moreover, his experimental model may be subject to an equation with periodic coefficients rather than one of the type (1) because the inertia of his pendulum varies harmonically with time. McLachlan [13] has given an example of a special equation of the type (1) in which $f(x)$ is strongly non-linear, and this equation was also given by Cartwright and Littlewood [1]. Moreover, there is a great deal of significant literature [6, 7, 9, 11, 16] exploring solutions of equations like (1). In an earlier paper [19] the origin of subharmonics of odd orders was discussed; however, that investigation contained many conjectures, and the treatment was re-

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†Numbers in square brackets refer to the bibliography at the end of the paper

stricted to odd orders. Here, we would like to lay a proper foundation for the earlier speculations, and to generalize the treatment to include the even-ordered subharmonics.

We shall be concerned only with periodic solutions of (1), and we use the following definitions. (I) If the Fourier expansion of the periodic solution of (1) contains a term $A_r \cos(\omega_r t + \varphi_r)$, where $\omega_r = \omega/r$ and r is an integer, and if $A_r \neq 0$, the solution is said to be *subharmonic* of order $1/r$. (II) If the solution is subharmonic of order $1/r$ it may be written as

$$x = A_r \cos\left(\frac{\omega}{r} t + \varphi_r\right) + \sum_{i \neq r} A_i \cos(\omega_i t + \varphi_i).$$

If in this solution $|A_r| \gg |A_i|$ for all i, the solution is said to be a *strong* subharmonic of order $1/r$. (III) If $A_r \neq 0$ and $A_i = 0$ for all i, the solution is called a *pure* subharmonic. (IV) If $A_r \neq 0$, $A_i = 0$ for all i, and $\varphi_r = 0$ as well, the solution is said to be a *simple* subharmonic.

In view of these definitions, the case of $r = \pm 1$ is not excluded. Therefore, the harmonic response is in the class of subharmonics; in fact, it is the subharmonic of order 1. Moreover, it is quite possible for a solution to contain Fourier components of several distinct subharmonic frequencies. Such solutions will be called *multiply* subharmonic.

Special interest centers on periodic solutions of (1) which are of period $2\pi r/\omega$ where r is an integer. They are subharmonic in the sense of our definition and arise when $A_i = 0$ for all $i < r$, except perhaps when $i = 0$. Such solutions may be written in the form

$$x = B_0 + \sum_{j=1,2,\cdots} B_j \cos\left(\frac{j\omega}{r} t + \varphi_j\right)$$

and they will be called *subharmonic steady states*. They are, in general, multiply subharmonic unless r is a prime number, and $B_0 = 0$. Evidently, the simple and pure subharmonics are subharmonic steady states.

The simple subharmonics. Under the restriction that $f(x)$ is analytic we will show that to every simple subharmonic

$$x = x_0 \cos\frac{\omega}{r} t, \tag{2}$$

where x_0 is a non-vanishing, real, finite but otherwise arbitrary constant, and r is an integer, there belongs one and only one equation

$$x^{\cdot\cdot} + f_r(x) = P_0 \cos \omega t \tag{3}$$

capable of producing it. Furthermore, the "elastic force" in that equation is of the form

$$f_r(x) = \sum_n^r \gamma_r^n x^n \tag{4}$$

and the coefficients γ_r^n, $(n = 1, 2, \cdots, r)$ are uniquely determined by the integer r and by the parameters of the equation. Moreover, any "equation of a simple subharmonic" (i.e., any equation possessing a simple subharmonic as a solution) is reducible to a one-parameter equation.

In view of (2) and a well-known trigonometric identity we have, when $r > 2$ is an integer*,

*Excluding negative integers is no restriction since the cosine is an even function.

$$\cos \omega t = 2^{r-1}\left(\frac{x}{x_0}\right)^r - \frac{r}{1!} 2^{r-3}\left(\frac{x}{x_0}\right)^{r-2} + \frac{r(r-3)}{2!} 2^{r-5}\left(\frac{x}{x_0}\right)^{r-4}$$
$$- \frac{r(r-4)(r-5)}{3!} 2^{r-7}\left(\frac{x}{x_0}\right)^{r-6} + \frac{r(r-5)(r-6)(r-7)}{4!} 2^{r-9}\left(\frac{x}{x_0}\right)^{r-8} - \cdots$$

with the series terminating when one of the coefficients vanishes. For $r = 1$ and $r = 2$ we have, respectively, $\cos \omega t = x/x_0$ and $\cos \omega t = 2(x/x_0)^2 - 1$. Since $x^{\cdot\cdot} = -(\omega/r)^2 x$ we find for the elastic force, when $r > 2$,

$$f_r(x) = \frac{x_0\omega^2}{r^2}\left(\frac{x}{x_0}\right) + P_0\left[2^{r-1}\left(\frac{x}{x_0}\right)^r - \frac{r}{1!} 2^{r-3}\left(\frac{x}{x_0}\right)^{r-2} + \frac{r(r-3)}{2!}\left(\frac{x}{x_0}\right)^{r-4}\right.$$
$$\left. - \frac{r(r-4)(r-5)}{3!} 2^{r-7}\left(\frac{x}{x_0}\right)^{r-6} + \frac{r(r-5)(r-6)(r-7)}{4!} 2^{r-9}\left(\frac{x}{x_0}\right)^{r-8} - \cdots\right]. \tag{5}$$

When $r = 1$, $f_1(x) = (x_0\omega^2 + P_0)(x/x_0)$ and when $r = 2$,

$$f_2(x) = \frac{x_0\omega^2}{4}\left(\frac{x}{x_0}\right) + P_0\left[2\left(\frac{x}{x_0}\right)^2 - 1\right].$$

It will be seen that the coefficient of P_0 is in every case a polynomial of degree r containing only even, or only odd, powers according as r is an even or an odd integer. For every positive integer r, the elastic force is also a power series of degree r. When r is odd we have

$$f_r(x) = \frac{x_0\omega^2}{r^2}\left(\frac{x}{x_0}\right) + P_0 \sum_{n=1,3,\cdots}^{r} \alpha_r^n\left(\frac{x}{x_0}\right)^n \tag{6}$$

and when r is even,

$$f_r(x) = \frac{x_0\omega^2}{r^2}\left(\frac{x}{x_0}\right) + P_0 \sum_{n=0,2,4,\cdots}^{r} \alpha_r^n\left(\frac{x}{x_0}\right)^n, \tag{7}$$

where the α_r^n depend on r only and are defined in an obvious manner through (5). It is an interesting fact that $f_r(x)$ is an odd power series when r is odd, but it is not even for even r.

We have shown that the set of equations having simple subharmonic solutions of orders $1/r$ are

$$x^{\cdot\cdot} + \frac{x_0\omega^2}{r^2} + P_0 \sum_n^r \alpha_r^n\left(\frac{x}{x_0}\right)^n = P_0 \cos \omega t, \qquad (r = 1, 2, \cdots). \tag{8}$$

If we set $\omega t = \tau$, $x/x_0 = \xi$ and $P_0/(x_0\omega^2) = k$, the simple subharmonics become $\xi = \cos \tau/r$, $(r = 1, 2, \cdots)$ and (8) becomes

$$\xi^{\cdot\cdot} + r^{-2}\xi + k \sum_n^r \alpha_r^n \xi^n = k \cos \tau, \qquad (r = 1, 2, \cdots). \tag{9}$$

These equations contain only the single parameter k. In view of the definition of k and the restrictions on P_0, x_0 and ω it is evident that all finite, non-zero values of k are admitted, but $k = 0$ is excluded. In Table 1 we tabulate the coefficients α_r^n for the equations of the simple subharmonics of orders $1/r$, $(r = 1, 2, \cdots, 9)$.

TABLE 1

r	α_r^0	α_r^1	α_r^2	α_r^3	α_r^4	α_r^5	α_r^6	α_r^7	α_r^8	α_r^9
1	0	1	0	0	0	0	0	0	0	0
2	−1	0	2	0	0	0	0	0	0	0
3	0	−3	0	4	0	0	0	0	0	0
4	1	0	−8	0	8	0	0	0	0	0
5	0	5	0	−20	0	16	0	0	0	0
6	−1	0	18	0	−48	0	32	0	0	0
7	0	−7	0	56	0	−112	0	64	0	0
8	1	0	−32	0	160	0	−288	0	128	0
9	0	9	0	−120	0	432	0	−576	0	256

The α-coefficients in Table 1 may be constructed from the following relations:

$$\begin{aligned}
\alpha_m^n &= 0 \quad \text{for all} \quad n > m \\
&= 0 \qquad\quad \text{when} \quad m + n = 2p - 1, \qquad (p = 1, 2, \cdots) \\
\alpha_r^0 &= 0 \qquad\quad \text{when} \quad \left.\begin{aligned} r &= 2p - 1 \\ r &= 4p - 2 \\ r &= 4p - 4 \end{aligned}\right\} (p = 1, 2, \cdots) \\
&= -1 \qquad\ \text{when} \\
&= +1 \qquad\ \text{when} \\
\alpha_r^1 &= r(-)^{(r-1)/2} \ \text{when} \quad r = 2p - 1, \qquad (p = 1, 2, \cdots)
\end{aligned}$$

$$\alpha_p^p = 2\alpha_{p-1}^{p-1}, \qquad (p = 2, 3, \cdots)$$

$$\begin{aligned}
\alpha_m^n = 2\alpha_{m-1}^{n-1} - \alpha_{m-2}^n \quad &\text{when} \quad m \geq n + 2 \\
&\text{when} \quad m + n \neq 2p - 1, \qquad (p = 1, 2, \cdots)
\end{aligned}$$

Finally, we observe that it is evident from the construction of the equation of the simple subharmonic $\xi = \cos r/r_0$ (where r_0 is a positive integer) that, if $f(\xi)$ is analytic, there exists one and only one equation of the form $\xi^{\cdot\cdot} + f(\xi) = k \cos \tau$ capable of producing it; this is the equation $\xi^{\cdot\cdot} + f_{r_0}(\xi) = k \cos \tau$, where

$$f_{r_0}(\xi) = r_0^{-2} + k \sum_n^{r_0} \alpha_{r_0}^n \xi^n$$

and the $\alpha_{r_0}^n$ are defined above.

We would like to discuss briefly the significance of the equations of the simple subharmonics.

If we are asked to find the "steady-state" periodic solutions (if any) of $x^{\cdot\cdot} + \sum_n \alpha^n x^n = P_o \cos \omega t$, we can do this by straightforward, elementary means only if $\sum_n \alpha^n x^n = \alpha^1 x$. For that case we find $x = x_0 \cos \omega t$, where $x_0 = P_0/(\alpha^1 - \omega^2)$, $[(\alpha^1)^{1/2} \neq \pm\omega]$ Now, this last relation is an equation of the sort $g(P_0, \alpha^1, \omega, x_0) =$ const., solved for the amplitude x_0. The method of obtaining this important relation involves techniques which fail unless $\alpha^2 = \alpha^3 = \cdots = 0$. In fact, the failure of these techniques in every but the linear case, presents one of the fundamental difficulties in the discussion of non-linear equations.

There is, however, another obvious way of finding the relation $g(P_0, \alpha^1, \omega, x_0) =$

const., and this way can be taken in the non-linear as well as the linear case. If we are asking for that function $f_1(x)$ for which $x^{\cdot\cdot} + f_1(x) = P_0 \cos \omega t$ has the solution $x = x_0 \cos \omega t$ we find by direct substitution that $f_1(x) = \alpha^1 x$, where $\alpha^1 = (P_0 + \omega^2 x_0)/x_0$; but this last is precisely the equation $g =$ const. solved here for α^1 instead of x_0. If we make the transformation from t to τ, from x to ξ, and put $P_0/(x_0\omega^2) = k$, the desired solution becomes $\xi = \cos \tau$, $f_1(\xi) = \alpha^1\xi$, and $\alpha^1 = 1 + k$. In view of the arbitrariness of x_0, P_0 and ω, this latter result preserves the generality of the former. As a consequence of this method of establishing a correspondence between equations of the form $\xi^{\cdot\cdot} + f(\xi) = k \cos \tau$ (with $f(\xi)$ analytic) and solutions $\xi = \cos \tau/r$ (with r and integer) we find that the *linear*, sinusoidally driven oscillator equation is not the only one having a "steady-state" sinusoidal response; instead, there is an infinite set of such equations. They are the "equations of the simple subharmonics," and the linear equation is simply that of the simple subharmonic of order 1.

It may be disappointing to those who cherish general solutions to equations with arbitrary parameters that the equations of the simple subharmonics have very special coefficients. We share this disappointment. However, a moment of reflection shows that these "special coefficients" actually define the relations which must exist between the parameters of the equations and the amplitude and frequency of the "steady-state" response. In this sense, these equations are no more special than the linear oscillator equation under simple harmonic excitation. Even the technique of prescribing amplitude and frequency of the solution before obtaining additional results is familiar in non-linear vibration problems. For instance, when searching for the harmonic response of the non-linear harmonically driven oscillator equation, the relation which exists between the amplitude of the response (of prescribed frequency!) on the one hand, and the frequency of the excitation on the other is normally obtained by prescribing the amplitude and computing the frequency [20]. Here, we have merely gone one step further and have inserted the governing relations in the equations of motion.

Stability of the simple subharmonics. It is well known that the stability of the periodic solution $\xi = \cos \tau/r$ of (8) depends on the stability of solutions of a Hill equation. In the case of the simple subharmonics, the stability investigation gains from discussing the odd and even subharmonics separately. Accordingly, we begin with an examination of the stability of the simple, *odd* subharmonics.

The stability of the simple subharmonic $\xi = \cos \tau/r$ of the equation

$$\xi^{\cdot\cdot} + r^{-2}\xi + k \sum_{n=1,3,\cdots}^{r} \alpha_r^n \xi^n = k \cos \tau, \qquad (r = 1, 3, \cdots)$$

is identified with the stability of solutions of

$$u^{\cdot\cdot} + \left\{r^{-2} + k\left[r(-)^{(r-1)/2} + \sum_{n=3,5,\cdots}^{r} n\alpha_r^n \cos^{n-1} \tau/r\right]\right\}u = 0. \tag{10}$$

In this equation, the term belonging to $n = 1$ has been written ahead of the summation, and use has been made of the relation $\alpha_r^1 = r(-)^{(r-1)/2}$ for all odd r.

When $m \geq 2$ is an even integer, we may write

$$\cos^m \frac{\tau}{r} = 2^{-(m-1)}\left\{\frac{1}{2}\binom{m}{\frac{m}{2}} + \sum_{i=0,1,2,\cdots}^{(m-2)/2} \binom{m}{i} \cos (m - 2i)\frac{\tau}{r}\right\}$$

and when $\bar{m} \geq 3$ is an odd integer we have

$$\cos^{\bar{m}} \frac{\tau}{r} = 2^{-(\bar{m}-1)} \sum_{i=0,1,2,\cdots}^{(\bar{m}-1)/2} \binom{\bar{m}}{i} \cos (\bar{m} - 2i) \frac{\tau}{r},$$

where the conventional notation for the binomial coefficients has been employed. Application of the first of the above relations reduces the stability problem of the odd simple subharmonics to a discussion of the Hill equation

$$u^{\cdot\cdot} + \left(r^{-2} + k\left\{r(-)^{(r-1)/2} + \sum_{n=3,5,\cdots}^{r} \frac{n\alpha_r^n}{2^{n-2}} \left[\frac{1}{2}\binom{n-1}{\frac{n-1}{2}} + \sum_{i=0,1,2,\cdots}^{(n-3)/2} \binom{n-1}{i} \cos (n - 1 - 2i) \frac{\tau}{r}\right]\right\}\right)u = 0. \tag{11}$$

A quantitative discussion of the stability of this equation is not possible in view of the complexity of the periodic coefficient. However, there exists evidence [22] that under certain circumstances the stability of the solutions of (11) is not highly sensitive to the precise form of the periodic coefficient. Accordingly, we shall adopt the viewpoint that the coefficient of u is a periodic function which, for purposes of a stability analysis, can be adequately represented by the leading terms of its Fourier expansion. One of the circumstances which must be met is, obviously, that the leading term must have the largest coefficient. If we write the second sum in (11) in an ascending order of Fourier terms we have

$$\sum_{i=0,1,2,\cdots}^{(n-3)/2} \binom{n-1}{i} \cos (n - 1 - 2i) \frac{\tau}{r} = \binom{n-1}{\frac{n-3}{2}} \cos \frac{2\tau}{r} + \binom{n-1}{\frac{n-5}{2}} \cos \frac{4\tau}{r} + \cdots + \cos \frac{(n-1)\tau}{r}.$$

In this sum, the magnitude of the coefficients decreases continually—i.e.,

$$\left| \binom{n-1}{i} \right| > \left| \binom{n-1}{i-1} \right|$$

for all integers i in $0 \leq i \leq (n - 3)/2$. If only the leading term of the Fourier expansion of

$$\sum_{i=0,1,2,\cdots}^{(n-3)/2} \binom{n-1}{i} \cos (n - 1 - 2i) \frac{\tau}{r}$$

is retained, the stability of the odd simple subharmonics depends on that of the solutions of

$$\frac{d^2u}{dz^2} + (a_0^r + b_0^r \cos z)u = 0,$$

where

$$\begin{aligned} a_0^r &= \frac{1}{4} + \frac{kr^2}{4}\left[r(-)^{(r-1)/2} + \sum_{n=3,5,\cdots}^{r} \frac{n\alpha_r^n}{2^{n-1}} \binom{n-1}{\frac{n-1}{2}}\right] \\ b_0^r &= \frac{kr^2}{2} \sum_{n=3,5,\cdots}^{r} \frac{n\alpha_r^n}{2^{n-1}} \binom{n-1}{\frac{n-3}{2}}. \end{aligned} \tag{12}$$

Now it can be shown that

$$\left.\begin{aligned}\sum_{n=3,5,\cdots}^{r} \frac{n\alpha_r^n}{2^{n-1}} \binom{n-1}{\frac{n-1}{2}} &= r \quad \text{when} \quad r = 4m - 1 \\ &= 0 \quad \text{when} \quad r = 4m - 3\end{aligned}\right\} \qquad (m = 1, 2, \cdots)$$

$$\sum_{n=3,5,\cdots}^{r} \frac{n\alpha_r^n}{2^{n-1}} \binom{n-1}{\frac{n-3}{2}} = r.$$

As a consequence, the coefficients in (12) become simply

$$\left.\begin{aligned} a_0^r &= \frac{1}{4}(1 - r^3k) \\ b_0^r &= \frac{1}{2} r^3 k \end{aligned}\right\} \qquad (r = 3, 5, \cdots). \tag{13}$$

The elimination of the parameter k between them yields for all odd integers $r \geq 3$ the relation

$$a_0^r = \frac{1}{4} + \frac{1}{2} b_0^r . \tag{14}$$

We call (14) the "stability characteristic" of the simple odd subharmonics of order $1/r$. It has the remarkable property of being independent of the order $1/r$. This stability characteristic is shown together with the stable and unstable domains of the ab-plane in Fig. 1. From it, we see that simple odd subharmonics have a stable range when

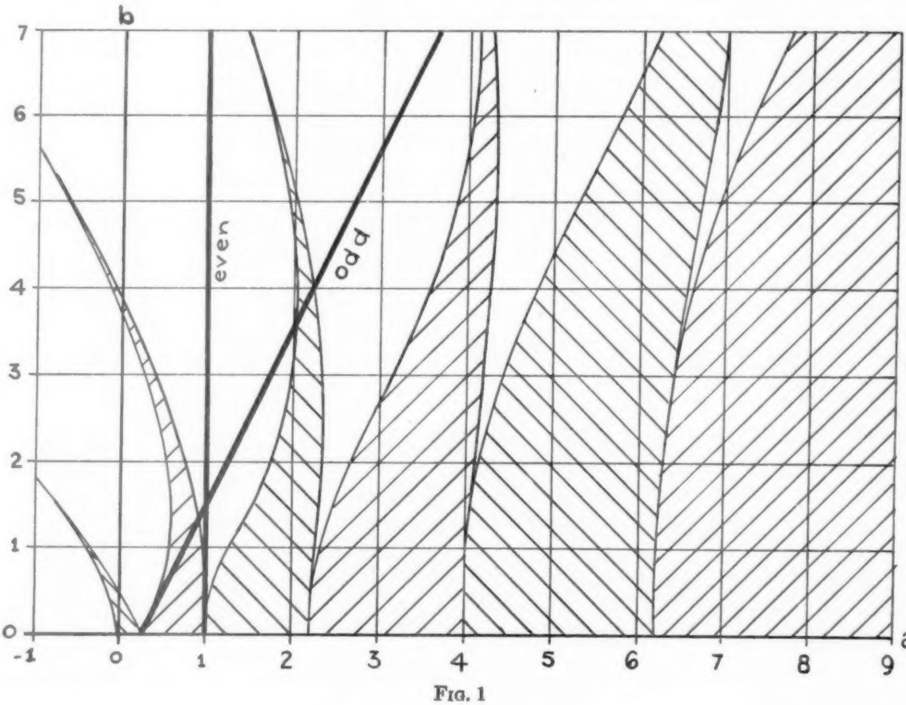

FIG. 1

$1/4 < a_0^r < 1$ and $0 < b_0^r < 3/2$ where the upper bounds are somewhat larger than the least upper bounds. As a_0^r and b_0^r increase, there follows a fairly extensive unstable range for $1 < a_0^r < 2$ and $3/2 < b_0^r < 5/2$, but near $a_0^r = 2$, $b_0^r = 5/2$ a very narrow stable range exists. This is again followed by an extensive unstable range and an extremely narrow stable one near $a_0^r = 4$, $b_0^r = 15/2$.

One of the interesting features of this result is that the simple odd subharmonics are never stable when k is negative. In fact the bounds on k which insure stability (in the first stability range) are $0 < k < 3/r^3$ where, again, the upper bound exceeds the least upper bound somewhat. In the case $r = 3$, for instance, positive k implies a hard spring, negative k a soft spring;* it is seen then that simple subharmonics of order 1/3 can not occur in a system with a soft spring.

Another interesting result is that the simple subharmonic may be stable when the equilibrium position $\xi = 0$ is not. For instance, in the case $r = 3$, the equilibrium position becomes unstable for $k > 1/27$ while the subharmonic is stable until $k = 3/27$, nearly.

We examine, now the stability of the *even* simple subharmonics. When a procedure is followed exactly analogous to the preceeding one, except that r is even, it is found that the stability of the even simple subharmonics is identified with the stability of solution of the equation

$$\frac{d^2u}{dz^2} + (\bar{a}_0^r + \bar{b}_0^r \cos z)u = 0$$

and the coefficients are

$$\bar{a}_0^r = 1, \qquad \bar{b}_0^r = 2r^3k. \tag{15}$$

It is seen that the stability characteristic of the even simple subharmonics is a straight line parallel to the b-axis and crossing the a-axis at $a = 1$. From Fig. 1, where this stability characteristic is shown, we deduce that the even simple subharmonics are almost never stable except in some very narrow ranges the first of which occurs near $|\bar{b}_0^r| = 8$ when $k = \pm 4/r^3$, approximately.

The strong subharmonics. In this section we inquire into the properties of solutions of equations which "lie in the neighborhood" of those of the simple subharmonics. Specifically, we consider the equation

$$\xi^{\cdot\cdot} = g(\tau, \xi, |\mu| \xi^{\cdot}, \mu) = g(\tau + T, \xi, |\mu| \xi^{\cdot}, \mu), \tag{16}$$

with $|\mu|$ small and $g(\tau, \xi, 0, 0) = -f_r(\xi) + k \cos \tau$. The function g is analytic in all variables. When $\mu = 0$, (16) has the solution $\xi = \xi_0(\tau) = \xi_0(\tau + T)$, but T need not be the least period of ξ_0 or g. When μ does not vanish, (16) has the solution $\xi = \eta(\tau, \mu)$. It can, then, be shown [2] that (16) has a solution $\xi = \eta(\tau, \mu)$ which is periodic in τ of period T and analytic in both arguments. Moreover, there is only one such solution for each μ. It should be noted that (16) includes as a special case the equation $\xi^{\cdot\cdot} + |\mu| \xi^{\cdot} + f_r(\xi) = k \cos \tau$, so that it has been shown that small positive damping does not have a profound effect on solutions obtained when damping is neglected.

Since $\eta(\tau, \mu)$ is analytic in the small parameter μ one may write

$$\eta(\tau, \mu) = \eta_0(\tau, 0) + \sum_i \mu^i \eta_i(\tau, 0) = \xi_0(\tau) + \sum_i \mu^i \xi_i(\tau),$$

*In hard springs, the stiffness increases with deflection; in soft springs it decreases with deflection.

where every ξ_i must be periodic since η is periodic. Thus, we have

$$\eta(\tau, \mu) = \cos \frac{\tau}{r} + \sum_i \sum_j \mu^i \left(a_{ij} \cos \frac{j\tau}{r} + b_{ij} \sin \frac{j\tau}{r} \right) \tag{17}$$

and, in view of the earlier definition, $\eta(\tau, \mu)$ is in general a strong subharmonic steady state.

As a special case of an odd strong subharmonic we consider a generalization of Duffing's equation

$$\xi^{\cdot\cdot} + r^{-2}\xi + k \sum_{n=1,3,\cdots}^{r} \alpha_r^n \xi^n + \mu\xi^{r+2} = k \cos \tau, \qquad (r = 1, 3, \cdots ,), \quad |\mu| \text{ small.} \tag{18}$$

It will be noticed that this reduces to Duffing's equation when $r = 1$.

When the theory of (17) is applied to (18), the solution to the latter, within linear terms in μ, turns out to be

$$\xi = \left[1 - \mu \binom{r+2}{\frac{r+1}{2}} \Big/ 2^{r+2}kr \right] \cos \frac{\tau}{r} + \mu \sum_{j=3,5,\cdots}^{r+2} a_{1j} \cos \frac{j\tau}{r}, \tag{19}$$

where for $j = 3, 5, \cdots, \leq (n-1)/2$

$$a_{1i} = -\binom{r+2}{\frac{r+2-j}{2}} \Big/ \left\{ 2^{r+1} \left[(1 - j^2)r^{-2} + k\left(r + \sum_{n=3,5,\cdots}^{r} \frac{n\alpha_r^n}{2^{n-1}} \binom{n-1}{\frac{n-1}{2} - j} \right) \right] \right\}$$

and for $j > (n-1)/2, \cdots, r-2, r, r+2$

$$a_{1j} = -\binom{r+2}{\frac{r+2-j}{2}} \Big/ \{2^{r+1}[(1 - j^2)r^{-2} + kr]\},$$

where the usual notation for the binomial coefficients has been used. The solution (19) is in general multiply subharmonic. Suppose $r = r_0$ to be odd integer which is not a prime. Let it have factors $j_0, j_1, \cdots, j_m$ such that the ratios $r_0/j_{01}, \cdots, r_0/j_m$ are integers $c_0, \cdots, c_m$. Then (19) becomes

$$\xi = \left[1 - \mu \binom{r_0+2}{\frac{r_0+1}{2}} \Big/ 2^{r_0+2}kr_0 \right] \cos \frac{\tau}{r_0} + \mu \sum_{i=0}^{m} a_{1j_i} \cos \frac{\tau}{c_i} + \mu \sum_{j \neq j_i} a_{1j} \cos \frac{j\tau}{r_0}, \tag{20}$$

where the first term is strongly subharmonic of order $1/r_0$, the terms of the first sum are subharmonic of orders $1/c_i$, $(i = 0, 1, \cdots, m)$ and the terms in the second sum are not subharmonic. This explains why one can find subharmonics of many orders in the solution of a non-linear oscillator equation.

The stability of the solution (19) depends on that of a Hill equation. When only the leading terms of the periodic coefficient are retained, the "stability equation" becomes

$$\frac{d^2u}{dz^2} + [(a_0^r + \mu a_1^r) + (b_0^r + \mu b_1^r) \cos z]u = 0,$$

where a_0^r and b_0^r are given in (13), and

$$a_1^r = \left[r^2\binom{r+2}{\frac{r+1}{2}}\Big/2^{r+2}\right]\{(n_0+1)[1-(-)^{(r-1)/2}]/4+(r+2)\},$$

$$b_1^r = \left[r^2\binom{r+2}{\frac{r+1}{2}}\Big/2^{r+2}\right]\left\{(n_1-4)/4+\binom{r+2}{\frac{r-1}{2}}(r+2)\right\},$$

$$n_0 = \sum_{n=3,5,\cdots}^{r}\frac{n^2\alpha_r^n}{2^{n-1}}\binom{n-1}{\frac{n-1}{2}}\Big/\{r[1-(-)^{(r-1)/2}]\},$$

$$n_1 = \sum_{n=3,5,\cdots}^{r}\frac{n^2\alpha_r^n}{2^{n-1}}\binom{n-1}{\frac{n-3}{2}}\Big/r.$$

It can be shown easily that n_0 and n_1 as well as a_1^r and b_1^r are positive for every r. The relation between a_1^r and b_1^r becomes a little more transparent when one writes

$$\binom{r+2}{\frac{r+1}{2}} = \binom{r+2}{\frac{r-1}{2}}\kappa(r)$$

in the expression for b_1^r . Since

$$\binom{r+2}{\frac{r+1}{2}} \geq \binom{r+2}{\frac{r-1}{2}}$$

the quantity $\kappa(r) \geq 1$. In fact, $\kappa(3) = 2$ and $\kappa(r) \to 1$ monotonically as $r \to \infty$. We now have

$$b_1^r = \left[r^2\binom{r+2}{\frac{r+1}{2}}\Big/2^{r+2}\right]\{(n_1-4)/4+\kappa(r)^{-1}(r+2)\}$$

from which it is seen by comparing it with the expression for a_1^r that $a_1^r/b_1^r \to 1$ as r increases. For the likely conjecture that, $n_0 = n_1 = r/2$ approximately, the relation between a_1^r and b_1^r may be expected to be as shown in Fig. 2. It follows, that the stability characteristic of (18) lies everywhere in the neighborhood of the "odd stability characteristic" of Fig. 1 but has a slightly different intercept with the a-axis. As a consequence, *stable* strong subharmonic steady states can occur for negative k. In particular, when $r = 3$, *strong* subharmonics may be stable in systems with soft springs while *simple* subharmonics are not, as shown earlier.

When the case of even-ordered subharmonics, analogous to the generalized Duffing equation is examined the equation may be written as

$$\ddot{\xi} + r^{-2}\xi + k(-)^{r/2} + k\sum_{n=2,4,\cdots}^{r}\alpha_r^n\xi^n + \mu\xi^{r+2} = k\cos\tau, \qquad (r = 2, 4, \cdots) \tag{21}$$

and the solution within linear terms in μ becomes

$$\xi = -\mu \frac{r^2}{2^{r+1}} \binom{r+2}{\frac{r+2}{2}} + \cos\frac{\tau}{r} + \mu \sum_{j=2,4,\cdots}^{r+2} a_{1j} \cos\frac{j\tau}{r}, \tag{22}$$

where

$$a_{1j} = -\binom{r+2}{\frac{r+2-j}{2}} \Big/ \{2^{r+1}[(1-j^2)r^{-2}]\}.$$

The stability of this solution depends (within the earlier simplification) on the solutions of

$$\frac{d^2u}{dz^2} + [(\bar{a}_0^r + \mu\bar{a}_1^r) + (\bar{b}_0^r + \mu\bar{b}_1^r)\cos z]u = 0,$$

where $\bar{a}_0^r$ and $\bar{b}_0^r$ are given in (15) and

$$\bar{a}_1^r = r^2 a_{10}\left[k \sum_{n=2,4,\cdots}^{r} \frac{n(n-1)\alpha_r^n}{2^{n-2}} \binom{n-2}{\frac{n-2}{2}} + \frac{r+2}{2^r}\binom{r}{r/2}\right],$$

$$\bar{b}_1^r = r^2 a_{10} \frac{r+2}{2^r}\binom{r+1}{r/2},$$

$$a_{10} = (r^2/2^{r+1})\binom{r+2}{\frac{r+2}{2}}.$$

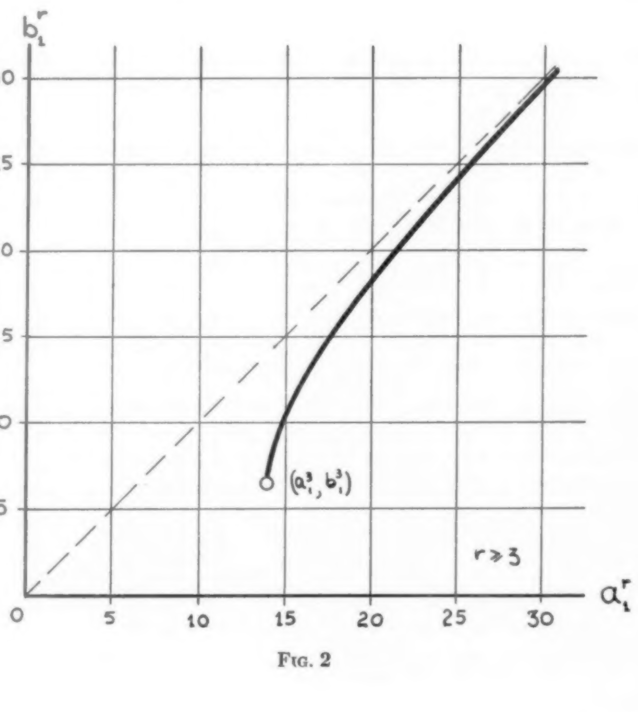

Fig. 2

The effect of these quantities on the stability can be established from the observation that $\bar{a}_1^r \neq 0$. Since $|\mu|$ is small, the stability characteristic will lie near the "even stability characteristic" of Fig. 1. However, it crosses the a-axis at $a = 1 + \mu\bar{a}_1^r$. Whether $\mu\bar{a}_1^r$ is positive or negative, the characteristic will pass through a stable range of the ab-plane for some small values of k in the interval $|k| > 0$. Thus, while *simple* even subharmonics are almost never stable, *strong* even subharmonics are easily stable.

Concluding remarks. We have concerned ourselves with the stable (and unstable) state of systems whose equations are special cases of

$$\xi^{\cdot\cdot} = F(\xi, \xi^{\cdot}, \mu, \tau) = F(\xi, \xi^{\cdot}, \mu, \tau + T),$$

where F is analytic in all variables on any interval, but in general not linear in them. Trefftz has remarked [21] that stable solutions tend with $\tau \to \infty$ to periodic solutions of period rT where r is an integer. In this paper, these solutions are called subharmonic for every r.

In spite of significant progress toward a better understanding of this equation in its general form [6, 7, 9, 11], or when it is nearly linear [16, 18], one cannot in general predict which among a variety of possible periods the solution will exhibit. Nor is the "mechanism" of subharmonic vibrations understood. With the mechanism we mean what Stoker [20] calls "a plausible physical explanation." An understanding of the mechanism seems to us essential for a profound and satisfying understanding of the phenomenon.

The latter difficulty has been removed—at least in the case of the simple subharmonic steady states—by the formulation of the equations of the simple subharmonics. Evidently, if we are ready to believe that the equation of the simple subharmonic of order 1 is the only one capable of producing the simple harmonic response, we should be ready to accept a similar statement for the simple subharmonics of any order.

Moreover, the formulation of these equations has robbed the linear equation of the singular position which it is usually thought to occupy as the only one whose solution is purely sinusoidal, or as the only one whose steady state has a uniquely determined period. For, it appears that the equations of the simple subharmonics of any order $1/r$ have a steady state $\xi = \cos \tau/r$ of unique frequency $1/r$. (We stipulate here that the steady state remains unchanged when the argument τ/r is replaced by $\tau/r + 2\pi n$, $(n = 1, 2, \cdots, r - 1)$; i.e., when the phase of the response is shifted by n cycles of the excitation.) This result is deduced as follows: the conventional method of determining the steady states (exclusive of their stability characteristics) is to assume as the solution a periodic function in τ in the form of a Fourier series with undetermined coefficients. The coefficients are, then, fixed by the relations which are obtained when the solution is substituted in the equation [20]. If this technique is used on the equation of the simple subharmonic of order $1/r$, and the period T of the assumed solution is commensurate with, but larger than $2\pi r$,—i.e., $T/2\pi r > 1$ is a rational number—it will evidently turn out that all coefficients vanish except that of $\cos \tau/r$ whose coefficient is 1. When the period T is not commensurate with $2\pi r$ the solution is not periodic in τ since the coefficient of $\cos \tau/r$ does not vanish. It is, in fact, almost periodic. This result bears out the conjecture [3, 13], that subharmonics of order $1/r$ can occur in equations of the form $\xi^{\cdot\cdot} + f(\xi) = k \cos \tau$ only if $f(\xi)$ is a power series of degree r or higher.

Finally, the multiplicity of steady states in non-linear equations of the type under discussion appears to have lost some of its mystery. An equation of the form $\xi^{\cdot\cdot} + f(\xi) =$

$k \cos \tau$ either is, or it is not, that of a simple subharmonic of order $1/r_0$. If it is, the steady state appears to be unique. If it is not, the steady states are $1/r_0, 1/r_1, \cdots$, where the r_i are integers and $r_i > r_{i-1}$. However, the steady states of orders $1/r_1, 1/r_2, \cdots$, are now merely the *higher order terms of the Fourier development* of a solution of period $2\pi r_0$. This observation by itself is, however, not sufficient to explain the occurrence of a multiplicity of subharmonics. For instance, a solution $\sum_i A_i \cos i\tau/r$ is periodic of order $1/m > 1/r$ only if m is a factor of r and, if the $A_i = 0$ for all $i = 1, 2, \cdots, (r/m) - 1$. Now, it is obvious that distinct periodic solutions of one and the same equation have different initial conditions (since the equations considered here satisfy the conditions for existence and uniqueness of solutions when the initial conditions are prescribed.) Therefore, one can explain the existence of distinct periodic solutions by regarding the Fourier coefficients in the solution $\sum_i A_i \cos i\tau/r$ as functions of the initial conditions. Let the initial conditions of the $1/m$th subharmonic be denoted by $\xi_m(0)$, $\xi_m^{\cdot}(0)$ where m is a factor of r. Then the subharmonic of order $1/m$ will be observed if it is stable, and if

$$A_1(\xi_m^{\cdot}(0), \xi_m^{\cdot}(0)) = \cdots = A_{(r/m)-1}(\xi_m^{\cdot}(0), \xi_m^{\cdot}(0)) = 0 \quad \text{while} \quad A_{r/m}(\xi_m(0), \xi_m^{\cdot}(0) \neq 0.$$

This observation does not aid in the construction of distinct subharmonic steady states; however, it provides a satisfactory mechanism for explaining them.

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STABILITY LIMITS FOR A CLAMPED SPHERICAL SHELL SEGMENT UNDER UNIFORM PRESSURE*

BY

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Summary. An integration procedure for the differential equations for the finite deflections of clamped shallow spherical shells under uniform pressure is developed. Stability limits for the clamped shell are obtained for a range of the central height to thickness ratio from about 1 to 35. This serves to correct and extend previously known stability limits for this problem.

1. Introduction. The existing literature on the subject of spherical shells allowing finite deflections under uniform radial pressure or point load at the apex divides into two parts. One part represented by the work of von Karman and Tsien [12, 13, 15];** Friedrichs [4]; Yoshimura and Uemura [14, 16]; Mushtari and Surkin [6]; and Feodosev [3] involves a determination of buckling pressures by means of a minimization of a potential energy expression for the shell with respect to a special class of deflection functions. Because of the rather special form of the assumed deflections in these papers, it is difficult to compare these results with integrations of the non-linear equations which as is noted in [8] can be derived as the Euler equations of the variational problem to minimize the potential energy of the shell; and therefore whose integrals correspond to a minimization with respect to a completely general class of deflection functions.

The other part represented by the work of Biezeno [2], Kaplan and Fung [5], and Simons [9] is based on integrations of non-linear differential equations corresponding to those which are used in this paper for shallow spherical shells. However, since Biezeno integrated the equations after assuming special forms for the non-linear terms in the differential equations, it is difficult to decide what influence this has on the results.

Kaplan and Fung are able to get integrals of the non-linear equations, but unfortunately they are able to determine buckling pressures, stresses, and deflections only for very low shells where the deflection shapes are of a simple type. In this range, their results are correct as far as they have gone in the perturbation of the non-linear equations, but appreciable corrections are to be found in the higher perturbations even in this range.

Simons generalizes the power series method given by Way for flat plates (see [11], p. 338) to shallow spherical shells. Numerical results when compared with those of Kaplan and Fung for the clamped shell are found to differ considerably owing to the retention of only a few terms in the power series solution.

In this paper (and in [2, 5, 9]), the so-called "classical criterion" of buckling as distinguished from the "energy criterion" developed by von Karman and Tsien is applied to interpret the buckling phenomenon. In the "classical criterion", it is assumed that a given state of equllibrium becomes unstable when there are equilibrium positions

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**Numbers in brackets refer to references at end of paper.

infinitesimally near to that state of equilibrium for the same external load. Thus, it is a question of obtaining the pressure-deflection relations for a given problem and properly interpreting the buckling pressure according to the above criterion. It is found that the center deflection to pressure relation used in [2, 5, 9] to interpret buckling must be generalized by interpreting buckling from a maximum deflection (in general, away from the center) to pressure relation in order to reveal the buckling in the cases where the deflection modes get more involved.

It might be noted here that problems of finite axi-symmetric deflections of flat plates are included as a limiting case of the shallow shell, and thus the methods given in this paper carry over to these problems.

With a view to the application of high speed digital computing equipment, the basic approach in this paper has been to reduce the integration of the non-linear differential equations to the problem of solutions of algebraic equations by means of suitable sets of functions for the various cases. Thus, the rapidly increasing store of methods for applying computers to solving algebraic equations can be brought to bear on these problems.

2. Equations for shallow spherical shells. The equations for the finite deflections of shallow spherical shells under uniform radial pressure which form the basis of this analysis are derived by Reissner in [8] and are listed here for reference:

$$D/a(\beta'' + \beta'/\xi - \beta/\xi^2) = -\Psi + \tfrac{1}{2}\rho a^2\xi + \Psi\beta/\xi, \tag{1a}$$

$$1/Eha(\Psi'' + \Psi'/\xi - \Psi/\xi^2) = \beta - \tfrac{1}{2}\beta^2/\xi, \tag{1b}$$

$$aV = \tfrac{1}{2}\rho a^2\xi, \qquad aQ = -\Psi + \tfrac{1}{2}\rho a^2\xi + \Psi\beta/\xi, \tag{2a}$$

$$aN_\xi = \Psi/\xi, \qquad aN_\theta = \Psi', \tag{2b}$$

$$aM_\xi = D(\beta' + \nu\beta/\xi), \qquad aM_\theta = D(\beta/\xi + \nu\beta'), \tag{2c}$$

$$u = \xi/Eh(\Psi' - \nu\Psi/\xi), \qquad w = -a\int \beta d\xi, \tag{2d}$$

$$\Psi = a\xi H, \qquad D = \frac{Eh^3}{12(1 - \nu^2)} \tag{2e}$$

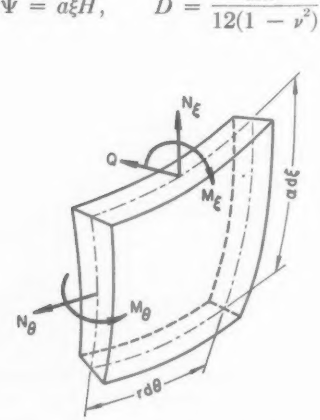

FIG. 1. Element of the shell showing stress resultants and couples.

The equations of the middle surface of the spherical shell in its undeformed state are taken in the form

$$r_0 = a \sin \xi \qquad z_0 = -a \cos \xi \qquad 0 \le \xi \le \xi_e ,$$

where ξ_e is to represent one-half the opening angle of the shell. The components of surface loading for uniform radial pressure take the form

$$p_h = \rho \sin \phi \qquad p_v = -\rho \cos \phi$$

Equations (1) and (2) are the result of restricting attention to shallow shells where $\xi_e \ll \pi/2$.

The following non-dimensional form of the variables will be used

$$\beta^* = \beta/\xi_e , \qquad \psi^* = \Psi/Eh\xi_e/m^2, \tag{3a}$$

$$p = -\rho/\rho_{cr} , \qquad x = \xi/\xi_e , \tag{3b}$$

$$\lambda^2 = \xi_e^2 m^2 a/h, \qquad m^4 = 12(1 - \nu^2), \tag{3c}$$

where ρ_{cr} is the minimum buckling pressure for the corresponding complete sphere from the linear theory ($\rho_{cr} = 4Eh^2/m^2a^2$, see [10]). Using the non-dimensional variables (1) becomes

$$1/\lambda^2 L^*\beta^* + \psi^* = -2px + \psi^*\beta^*/x, \tag{4a}$$

$$1/\lambda^2 L^*\psi^* - \beta^* = -\tfrac{1}{2}\beta^{*2}/x, \tag{4b}$$

where

$$L^*(\cdots) = (\cdots)'' + (\cdots)'/x - (\cdots)/x^2. \tag{5}$$

The corresponding expressions for stress resultants, stress couples, and displacements take the form

$$aN_\xi = Eh^2\psi^*/xm^2, \qquad aN_\theta = Eh^2\psi^{*\prime}/m^2, \tag{6a}$$

$$aM_\xi = D(\beta^{*1} + \nu\beta^*/x), \qquad aM_\theta = D(\beta^*/x + \nu\beta^{*1}), \tag{6b}$$

$$u/h = \xi_e/m^2(\psi^{*\prime} - \nu\psi^*/x)x, \qquad w/h = -\lambda^2/m^2 \int \beta^* \, dx. \tag{6c}$$

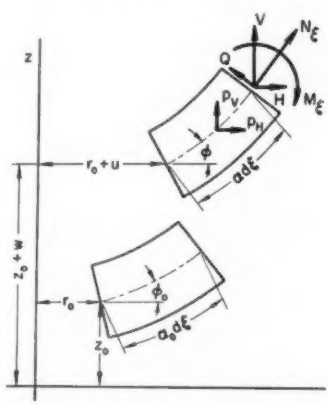

FIG. 2. Side view of element of shell in undeformed and deformed states.

3. Statement of the problem. The edge conditions for a clamped shell segment are given by

$$u(\xi_e) = 0, \qquad \beta(\xi_e) = 0, \tag{7}$$

which in non-dimensional form becomes using (6c)

$$\psi^{*\prime}(1) - \nu\psi^*(1) = 0, \qquad \beta^*(1) = 0. \tag{8}$$

In addition to (8) for shells without central hole we have the condition

$$\beta^*(x),\ \psi^*(x) \quad \text{regular at} \quad x = 0. \tag{9}$$

The small finite deflections of a clamped spherical segment under uniform radial pressure are determined by solutions of the Eq. (4) subject to the conditions (8) and (9).

4. Perturbation solution. A convenient method for getting solutions of (4) is to expand β^*, ψ^*, and the inward pressure p into series in powers of a certain parameter and convert (4) into a sequence of systems of linear differential equations. The perturbation parameter W will take the form of a ratio of deflection to thickness as a result of conditions imposed later. Thus, we write

$$\beta^* = \sum_{l=1}^{\infty} \beta_l W^l, \qquad \psi^* = \sum_{l=1}^{\infty} \psi_l W^l, \tag{10a}$$

$$2p = \sum_{l=1}^{\infty} p_l W^l \tag{10b}$$

and substitute into (4). Equating coefficients of powers of W to zero leads to the sequence of linear systems

$$1/\lambda^2 L^*\beta_1 + \psi_1 = -p_1 x, \tag{11.1a}$$

$$1/\lambda^2 L^*\psi_1 - \beta_1 = 0, \tag{11.1b}$$

$$1/\lambda^2 L^*\beta_2 + \psi_2 = -p_2 x + \beta_1\psi_1/x, \tag{11.2a}$$

$$1/\lambda^2 L^*\psi_2 - \beta_2 = -\tfrac{1}{2}\beta_1^2/x, \tag{11.2b}$$

$$\vdots$$

$$1/\lambda^2 L^*\beta_l + \psi_l = -p_l x + \sum_{i+j=l} \beta_i\psi_j/x, \qquad (i, j = 1, 2, \cdots, l-1) \tag{11$\cdot l$a}$$

$$1/\lambda^2 L^*\psi_l - \beta_l = -\tfrac{1}{2} \sum_{i+j=l} \beta_i\beta_j/x. \tag{11$\cdot l$b}$$

$$\vdots$$

The boundary conditions in terms of β_l and ψ_l become

$$\beta_l(x), \qquad \psi_l(x) \quad \text{regular at} \quad x = 0 \tag{12.la}$$

$$\beta_l(1) = 0, \qquad \psi_l'(1) - \nu\psi_l(1) = 0 \tag{12.lb}$$

for all l.

4.1. Determination of β_1, ψ_1, and p_1. β_1 and ψ_1 are to be solutions of (11.1) satisfying the conditions (12.1). We seek solutions of (11.1) as expansions in terms of Bessel

functions of the first kind in the form

$$\beta_1(x) = \sum_{n=1}^{\infty} a_n^{(1)} J_1(\lambda_n x), \tag{13a}$$

$$\psi_1(x) = c_1 x + \sum_{n=1}^{\infty} b_n^{(1)} J_1(\lambda_n x), \tag{13b}$$

where the λ_n are defined by

$$J_1(\lambda_n) = 0 \qquad (n = 1, 2, \cdots). \tag{14}$$

Thus, β_1 satisfies (12.1), and it remains to choose c_1 so that ψ_1 satisfies (12.1). From (13b), we have

$$\psi_1'(1) = c_1 + \sum_{n=1}^{\infty} \lambda_n b_n^{(1)} J_0(\lambda_n)$$

and

$$\nu\psi_1(1) = \nu c_1 ,$$

where we have used (14) and the formula

$$\frac{d}{dx} J_1(\lambda_n x) = \lambda_n J_0(\lambda_n x) - 1/x J_1(\lambda_n x). \tag{15}$$

Now ψ_1 will satisfy (12.1) if

$$c_1 = -\frac{1}{1-\nu} \sum_{n=1}^{\infty} \lambda_n b_n^{(1)} J_0(\lambda_n). \tag{16}$$

If (13) is substituted into (11.1), it follows that

$$-(\lambda_n/\lambda)^2 a_n^{(1)} + b_n^{(1)} = -(c_1 + p_1)\Gamma_n , \tag{17a}$$

$$a_n^{(1)} + (\lambda_n/\lambda)^2 b_n^{(1)} = 0, \tag{17b}$$

where we have used

$$L^*[J_1(\lambda_n x)] = -\lambda_n^2 J_1(\lambda_n x) \tag{18}$$

and equated coefficients of $J_1(\lambda_n x)$ to zero. The Γ_n are defined by

$$x = \sum_{n=1}^{\infty} \Gamma_n J_1(\lambda_n x) \tag{19}$$

and using

$$\int_0^1 x^2 J_1(\lambda_n x)\, dx = -1/\lambda_n J_0(\lambda_n) \tag{20a}$$

$$\int_0^1 x J_1^2(\lambda_n x)\, dx = \tfrac{1}{2} J_0^2(\lambda_n) \tag{20b}$$

we get

$$\Gamma_n = -\frac{2}{\lambda_n J_0(\lambda_n)} \tag{21}$$

Solving (17) gives

$$a_n^{(1)} = \frac{-2(c_1 + p_1)(\lambda_n/\lambda)^2}{\lambda_n J_0(\lambda_n)[1 + (\lambda_n/\lambda)^4]}, \tag{22a}$$

$$b_n^{(1)} = \frac{2(c_1 + p_1)}{\lambda_n J_0(\lambda_n)[1 + (\lambda_n/\lambda)^4]}. \tag{22b}$$

The coefficients p_s will be determined from the following conditions on the deflection. Using the expression for w/h from (6c), it follows that

$$\frac{w(x^*) - w(1)}{h} = -\frac{\lambda^2}{m^2}\int_1^{x^*} (\beta_1 W + \beta_2 W^2 + \cdots)\, dx, \tag{23}$$

where x^* $(0 \leq x^* \leq 1)$ will be taken at the point where the deflection is a maximum. If we impose the conditions

$$1 = \lambda^2/m^2 \int_{x^*}^1 \beta_1(x)\, dx, \tag{24.1}$$

$$0 = \int_{x^*}^1 \beta_2(x)\, dx, \tag{24.2}$$

$$\vdots$$

$$0 = \int_{x^*}^1 \beta_l(x)\, dx, \tag{24·l}$$

$$\vdots$$

then (23) reduce to

$$\frac{w(x^*) - w(1)}{h} = W \tag{25}$$

which serves to define the parameter W. The Eq. (24) will determine the coefficients p_l in the expansion for the inward pressure p.

To determine p_1, we use (13) to write (24.1) in the form

$$\begin{aligned} 1 &= \lambda^2/m^2 \int_{x^*}^1 \sum_{n=1}^{\infty} a_n^{(1)} J_1(\lambda_n x)\, dx \\ &= -\frac{2(c_1 + p_1)\lambda^2}{m^2} \sum_{n=1}^{\infty} \frac{(\lambda_n/\lambda)^2[J_0(\lambda_n x^*) - J_0(\lambda_n)]}{\lambda_n^2 J_0(\lambda_n)[1 + (\lambda_n/\lambda)^4]} \end{aligned}$$

or

$$-2(c_1 + p_1) = m^2\left\{ \sum_{n=1}^{\infty} \frac{J_0(\lambda_n x^*) - J_0(\lambda_n)}{J_0(\lambda_n)[1 + (\lambda_n/\lambda)^4]} \right\}, \tag{26}$$

where we have used the integral formula

$$\int J_1(\lambda_n x)\, dx = -\frac{1}{\lambda_n} J_0(\lambda_n x). \tag{27}$$

Since c_1 is known from (16), p_1 is determined.

4.2. Solution for any β_l, ψ_l, and p_l. Let

$$\beta_l(x) = \sum_{n=1}^{\infty} a_n^{(l)} J_1(\lambda_n x), \tag{28a}$$

$$\psi_l(x) = c_l x + \sum_{n=1}^{\infty} b_n^{(l)} J_1(\lambda_n x), \tag{28b}$$

then substitution into (11.l) leads to

$$-(\lambda_n/\lambda)^2 a_n^{(l)} + b_n^{(l)} = -(c_l + p_l)\Gamma_n + H_n^{(l)}, \tag{29a}$$

$$a_n^{(l)} + (\lambda_n/\lambda)^2 b_n^l = G_n^{(l)}, \tag{29b}$$

where the $G_n^{(l)}$, $H_n^{(l)}$ are defined by

$$G_n^{(l)} = \frac{1}{2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} C_n^{ij} \Big(\sum_{s+t=l} a_i^{(s)} a_j^{(t)} \Big), \qquad (s, t = 1, 2, \cdots, l-1) \tag{30a}$$

$$H_n^{(l)} = \sum_{s+t=l} c_s a_n^{(t)} + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} C_n^{ij} \Big(\sum_{s+t=l} a_i^{(s)} b_j^{(t)} \Big) \tag{30b}$$

and the C_n^{ij} defined by

$$1/x J_1(\lambda_i x) J_1(\lambda_j x) = \sum_{n=1}^{\infty} C_n^{ij} J_1(\lambda_n x). \tag{31}$$

The $G_n^{(l)}$ and $H_n^{(l)}$ are known from the $1, 2, \cdots, l-1$ stages of the computation. Solving (29), gives

$$a_n^{(l)} = \left\{ -\frac{2(c_l + p_l)(\lambda_n/\lambda)^2}{\lambda_n J_0(\lambda_n)} - \left(\frac{\lambda_n}{\lambda}\right)^2 H_n^{(l)} + G_n^{(l)} \right\} [1 + (\lambda_n/\lambda)^4]^{-1} \tag{32a}$$

$$b_n^{(l)} = \left\{ \frac{2(c_l + p_l)}{\lambda_n J_0(\lambda_n)} + H_n^{(l)} + (\lambda_n/\lambda)^2 G_n^{(l)} \right\} [1 + (\lambda_n/\lambda)^4]^{-1} \tag{32b}$$

and for the lth stage (16) becomes

$$c_l = -\frac{1}{1-\nu} \sum_{n=1}^{\infty} \lambda_n b_n^{(l)} J_0(\lambda_n) \tag{33}$$

which forces ψ_l to satisfy the boundary condition (12.l). The condition (24.l) leads to an equation for the lth stage similar to (26) for the first in the form

$$-2(c_l + p_l) = \lambda^2 \left\{ \sum_{n=1}^{\infty} \frac{J_0(\lambda_n x^*) - J_0(\lambda_n)}{J_0(\lambda_n)[1 + (\lambda_n/\lambda)^4]} \right\}^{-1}, \\ \cdot \left\{ \sum_{n=1}^{\infty} \frac{[(\lambda_n/\lambda)^2 H_n^{(l)} - G_n^{(l)}][J_0(\lambda_n x^*) - J_0(\lambda_n)]}{\lambda_n [1 + (\lambda_n/\lambda)^4]} \right\}. \tag{34}$$

Therefore, c_l and p_l are determined; and the $a_n^{(l)}$ and $b_n^{(l)}$ are completely determined.

Starting with $a_n^{(1)}$, $b_n^{(1)}$, c_1, and p_1; the $G_n^{(2)}$ and $H_n^{(2)}$ can be computed and then $a_n^{(2)}$, $b_n^{(2)}$, c_2, and p_2. Similarly, $G_n^{(l)}$ and $H_n^{(l)}$ are computed leading to $a_n^{(l)}$, $b_n^{(l)}$, c_l, and p_l. This can be continued to obtain any number of terms in the series for β^*, ψ^*, and $2p$. Stresses and displacements can be computed from the Eq. (6). The buckling pressure

is determined from

$$2p = \sum_{l=1}^{\infty} p_l W^l \tag{35}$$

by the condition

$$dp/dW = 0. \tag{36}$$

5. Numerical solutions. This section involves numerical calculations using the integration procedure set up in the previous section to obtain new results for the finite deflections of a clamped spherical shell segment under uniform pressure. The perturbation of the non-linear equations is carried out far enough to determine the deflection curves as a function of the pressure for maximum deflections up to about one thickness of the shell. For the case of inward pressure, the buckling phenomenon is observed in this range of deflections and buckling pressures are found using the "classical criterion" for a range of the central height to thickness ratio from about 1 to 35. (See Figs. 3 and 4).

In the numerical computation for this problem or for other cases of loading and edge restraint involving shallow spherical shells, it is necessary to compute the expansion coefficients, c_n^{ij}, which enter the computation from the non-linear terms. These were computed directly from the definition

$$C_n^{ij} = [\tfrac{1}{2}J_0^2(\lambda_n)]^{-1} \int_0^1 J_1(\lambda_i x) J_1(\lambda_j x) J_1(\lambda_n x)\, dx \tag{37}$$

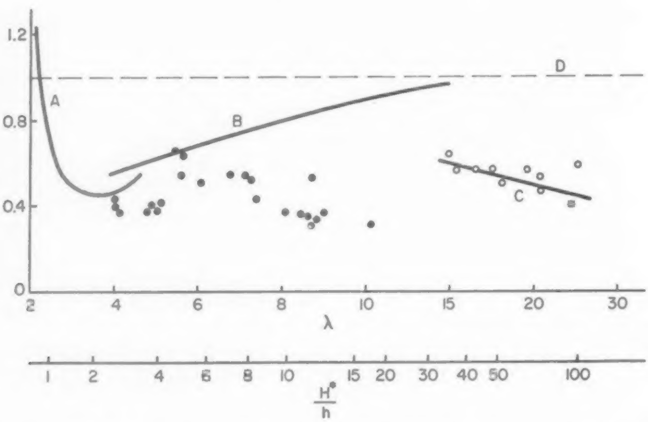

FIG. 3. Known theoretical and experimental results are compared. A represents the theory in [5], *B* the theory in this paper, *C* the theory in [12], the solid dots the experiments in [5], the hollow dots the experiments in [12], and *D* the classical buckling load for a complete sphere given here for reference.

using Simpson's rule with one hundred values of the integrand. The results obtained by machine computation are given in Table 1. It is shown in [1] that these coefficients can be used to solve a variety of other problems involving other conditions of loading and edge restraint.

In determining the buckling pressures given in Fig. 3, terms were computed in the perturbation series used in finding the pressure-maximum deflection curves until it was

seen that the higher order terms were no longer substantially affecting the curves in the neighborhood of the point where $dp/dW = 0$. The linear solution checks the linear solution given in [5] which is the same as the one given by Reissner in [7]. The second

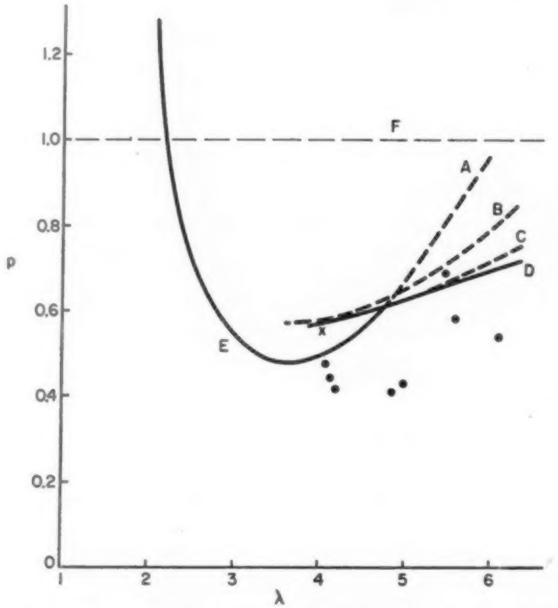

FIG. 4. The curves A, B, C, and D represent results using two, three, four, and five terms respectively in the perturbation series of this paper. E and the cross are computed in [5] using two and four terms respectively in a perturbation series while the solid dots represent the experimental results reported in [5]. F represents the buckling pressure for a complete sphere (linear theory) and is given here for reference.

TABLE 1a. *Numerical values for the expansion coefficients C_1^{ij} and C_2^{ij}.*

$C_1^{ij} \times 10$	1	2	3	4	5	6	7	8
1	10.123	3.0232	−0.20513	0.067211	−0.029812	0.015362	−0.0094689	0.0060496
2		5.9445	1.9771	−0.15236	0.054438	−0.025949	0.014360	−0.0089100
3			4.1592	1.4594	−0.11913	0.044558	−0.029812	0.012264
4				3.1925	1.1564	−0.097467	0.037596	−0.032196
5					2.5891	0.95663	−0.082360	0.032196
6						2.1770	0.81565	−0.071173
7							1.8778	0.71087
8								1.6506
$C_2^{ij} \times 10$								
1	5.4449	10.706	3.5608	−0.27440	0.098043	−0.046734	0.025862	−0.016047
2		8.1792	7.4106	2.4816	−0.21304	0.082514	−0.042530	0.023878
3			6.1991	5.5398	1.8982	−0.17083	0.069546	−0.036298
4				4.88463	4.4079	1.5366	−0.14285	0.058814
5					4.0093	3.6553	1.2906	−0.12160
6						3.3932	3.1208	1.1123
7							2.9386	2.7218
8								2.5901

TABLE 1b. *Numerical values for the expansion coefficients* C_3^{ij} *and* C_4^{ij}.

$C_3^{ij} \times 10$	1	2	3	4	5	6	7	8
1	−0.53368	5.1437	10.821	3.7968	−0.30992	0.11592	−0.05795	0.031905
2		10.705	8.9547	8.0023	2.7420	−0.24677	0.10046	−0.052433
3			9.0110	7.3074	6.2026	2.1426	−0.20310	0.085728
4				7.2969	5.9928	5.0455	1.7617	−0.16981
5					6.0597	5.0423	4.2489	1.4948
6						5.1395	4.3403	3.6684
7							4.4845	3.8044
8								3.9618

$C_4^{ij} \times 10$								
1	0.22866	−0.51834	4.9651	10.861	3.9340	−0.33159	0.12791	−0.063310
2		4.6872	10.465	9.2271	8.3266	2.9027	−0.26985	0.11110
3			9.5559	9.5422	7.8368	6.5981	2.3038	−0.22206
4				8.4625	8.0664	6.5892	5.4518	1.9135
5					7.2569	6.7884	5.6391	4.6415
6						6.2774	5.9621	4.9127
7							5.5059	5.2500
8								4.8913

TABLE 1c. *Numerical values for the expansion coefficients* C_5^{ij} *and* C_6^{ij}.

$C_5^{ij} \times 10^{+1}$	1	2	3	4	5	6	7	8
1	−0.12529	0.22878	−0.50064	4.8597	10.881	4.0204	−0.34613	0.13531
2		−0.49712	4.4294	10.286	9.3556	8.5295	3.0115	−0.28374
3			10.020	9.6808	9.7889	8.1454	6.8637	2.4148
4				9.9644	8.9645	8.4968	6.9660	5.7337
5					8.8285	7.9038	7.3757	6.0342
6						7.7507	6.9680	6.4774
7							6.8521	6.1939
8								6.1189

$C_6^{ij} \times 10^{+1}$								
1	0.07686	−0.12983	0.2293	−0.48764	4.7862	10.892	4.0808	−0.35609
2		0.22922	−0.47456	4.2687	10.154	9.4264	8.6695	3.0890
3			4.1205	9.7031	9.6969	9.9222	8.3468	7.0547
4				9.6901	10.115	9.2316	8.7678	7.2246
5					9.4093	9.2271	8.2952	7.[illegible]12
6						8.5468	8.2610	7.4116
7							7.6093	7.4078
8								6.9348

TABLE 1d. *Numerical values of the expansion coefficients* C_7^{ij} *and* C_8^{ij}.

$C_7^{ij} \times 10^{+1}$	1	2	3	4	5	6	7	8
1	−0.054954	0.083337	−0.12927	0.21819	−0.47798	4.7337	10.898	4.1255
2		−0.13705	0.22411	−0.46033	4.1587	10.057	9.4692	8.7706
3			−0.45308	3.9300	9.4784	9.6822	10.004	8.4868
4				9.3001	9.6196	10.171	9.3924	8.9559
5					10.185	9.6224	9.4624	8.5534
6						9.5827	8.9196	8.5929
7							8.7609	8.1406
8								7.9766

$C_8^{ij} \times 10^{+1}$

1	0.039952	−0.058843	0.080990	−0.12290	0.21263	−0.47003	4.6946	10.901
2		0.087558	−0.13310	0.21567	−0.44588	4.07865	9.9804	9.4976
3			0.21762	−0.43106	3.7946	9.3122	9.6574	10.057
4				3.7145	9.0100	9.5364	10.191	9.4949
5					9.4823	10.178	9.7333	9.6154
6						9.7833	9.7782	9.1540
7							9.2635	9.0768
8								8.5853

order solution checks with that given in [5] for $3 \leq \lambda \leq 5$ where the maximum deflection is at the center.

The Massachusetts Institute of Technology digital computer, Whirlwind I, was used extensively to reduce the computation time. The programs used for the machine were checked by independent calculations using a desk calculator.

In Figs. 3 and 4, the stability limits found in this paper are compared with known experimental and theoretical results. It is seen that the theoretical curve given in [5] based on two terms of a perturbation series is subject to considerable correction when more terms in the series are computed. In particular, the minimum value of H^*/h for which buckling occurs must be revised upward from the value of about 0.67 given in [5] to 2.2 found in this paper. Also, it is seen that the experimental results depart from the theoretical results of this investigation as H^*/h increases. As indicated in the references, this is probably due to the fact that the shell, under disturbances during the testing, jumps to a nearby buckles state before reaching the buckling pressure predicted by the theory in this paper which allows only continuous load-deflection processes.

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ON TRANSVERSE VIBRATIONS OF SHALLOW SPHERICAL SHELLS*

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1. Introduction. The present paper is concerned with the frequencies of free vibrations of shallow spherical shells of constant thickness. It has been shown earlier that for vibrations of shallow shells which are primarily transverse a considerable simplification of the problem can be effected by a justified neglect of longitudinal inertia in comparison with transverse inertia [1].

Previous applications of this observation included a study of axi-symmetrical vibrations of spherical shells [2], and a study of inextensional vibrations of general shallow shells [3]. In the present paper we investigate vibrations of shallow spherical shells, without axial symmetry. Appropriate solutions of the differential equations are obtained and these are used to obtain the frequencies of free vibrations of a spherical shell segment (or cap) with free edges, in their dependence on the curvature of the segment and on the number of nodal circles and diameters.

We may summarize certain qualitative aspects of our results as follows. Let H be the height of the apex of the spherical cap above the edge plane of the cap and let h be the wall thickness of the shell. When $H/h = 0$ we have Kirchhoff's results for the flat plate. When H/h tends to infinity the frequencies of free vibrations of the cap tend either to a limiting frequency which may be called membrane frequency or they tend to the frequencies of inextensional vibrations which we have previously considered [3]. The membrane frequency is limiting frequency for all vibrations with one or more nodal circles. Its value is independent of the number of nodal radii and of nodal circles provided the latter is not zero. On the other hand, the frequencies of inextensional vibrations are a function of the number of nodal radii and presuppose that the number of nodal circles is zero.

2. Differential equations for free transverse vibrations of shallow spherical shells. The differential equations which are to be solved are of the form

$$\nabla^2\nabla^2 F - \frac{C}{R}\nabla^2 w = 0, \tag{2.1}$$

$$D\nabla^2\nabla^2 w + \frac{1}{R}\nabla^2 F + \rho h\frac{\partial^2 w}{\partial t^2} = 0. \tag{2.2}$$

The various quantities occurring in (2.1) and (2.2) have the following significance

w = transverse (axial) displacement,
F = Airy's stress function,
R = radius of middle surface of shell,
h = wall thickness of shell,
C = Eh, longitudinal stiffness factor,
E = modulus of elasticity,

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$D \;\; = Eh^3/12\,(1 - \nu^2)$, bending stiffness factor,
$\nu \;\; =$ Poisson's ratio,
$\rho \;\; =$ density of shell material,
$\nabla^2 = (\;)_{,rr} + r^{-1}(\;)_{,r} + r^{-2}(\;)_{\theta\theta}$, Laplace operator in polar coordinates r, θ.

Stress resultants and couples, in polar coordinate form, are given by the following expressions.

$$\left.\begin{aligned} N_\theta = \frac{\partial^2 F}{\partial r^2}, \qquad N_r = \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}, \\ N_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial \theta}\right) \end{aligned}\right\}, \tag{2.3}$$

$$\left.\begin{aligned} V_r &= -D\frac{\partial \nabla^2 w}{\partial r} + \frac{1}{r}\frac{\partial M_{r\theta}}{\partial \theta} - \frac{r}{R}N_r \\ V_\theta &= -D\frac{\partial \nabla^2 w}{r\partial \theta} + \frac{\partial M_{r\theta}}{\partial r} - \frac{r}{R}N_{r\theta} \end{aligned}\right\}, \tag{2.4}$$

$$\left.\begin{aligned} M_r &= -D\left(\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\frac{\partial w}{\partial r} + \frac{\nu}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) \\ M_\theta &= -D\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \nu\frac{\partial^2 w}{\partial r^2}\right) \\ M_{r\theta} &= -(1-\nu)D\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right) \end{aligned}\right\} \tag{2.5}$$

Equations (2.1) to (2.5), except for the inertia term $\rho h w_{tt}$, are the same as those for problems of statics [4].

3. Solutions for simple harmonic motion. We consider solutions of the form

$$(w, F) = \begin{pmatrix}\cos n\theta \\ \sin n\theta\end{pmatrix}[w_n(r), F_n(r)]e^{ipt}. \tag{3.1}$$

Substitution of (3.1) into the differential equations (2.1) and (2.2) leads to the ordinary differential equations

$$L_n^2 F_n - \frac{C}{R}L_n w_n = 0, \tag{3.2}$$

$$DL_n^2 w_n - \rho h p^2 w_n + \frac{1}{R}L_n F_n = 0, \tag{3.3}$$

where the operator L_n is defined by

$$L_n = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}. \tag{3.4}$$

We may write the solutions of (3.2) and (3.3) as follows

$$w_n = \frac{1}{RD\lambda^4}\psi_n + \chi_n, \tag{3.5}$$

$$F_n = \left(1 + \frac{C}{R^2 D\lambda^4}\right)\psi_n^* + \frac{C}{R\lambda^4} L_n \chi_n + \phi_n . \tag{3.6}$$

In Eqs. (3.5) and (3.6) we have

$$\lambda = \left[\frac{1}{D}\left(h\rho p^2 - \frac{C}{R^2}\right)\right]^{1/4} \tag{3.7}$$

and

$$\psi_n = \begin{cases} C_{1,0} + C_{2,0} \log r, & n = 0 \\ C_{1,n} r^n + C_{2,n} r^{-n}, & 1 \le n \end{cases} \tag{3.8}$$

$$\phi_n = \begin{cases} C_{3,0} + C_{4,0} \log r, & n = 0 \\ C_{3,n} r^n + C_{4,n} r^{-n}, & 1 \le n \end{cases} \tag{3.9}$$

$$\psi_n^* = \begin{cases} \frac{1}{4} C_{1,0} r^2 + \frac{1}{4} C_{2,0} r^2 (\log r - 1), & n = 0 \\ \frac{1}{8} C_{1,1} r^3 + \frac{1}{2} C_{2,1} r \log r, & n = 1 \\ C_{1,n} \dfrac{r^{2+n}}{4(n+1)} + C_{2,n} \dfrac{r^{2-n}}{4(1-n)}, & 2 \le n \end{cases} \tag{3.10}$$

$$\chi_n = A_{1,n} J_n(\lambda r) + A_{2,n} Y_n(\lambda r) + A_{3,n} I_n(\lambda r) + A_{4,n} K_n(\lambda r). \tag{3.11}$$

The functions J_n, Y_n, I_n and K_n are Bessel functions and modified Bessel functions of order n. We note that the argument of these functions is real, as long as the frequency p is greater than a reference frequency which we denote by p_∞ and which is given by

$$p_\infty^2 = \frac{C}{h\rho R^2}. \tag{3.12}$$

Alternatively, p_∞ may be written in terms of the rise H of the shell segment which, from $z = r^2/2R$, follows in the form $H = a^2/2R$, where a is the base radius of the shell segment, as

$$p_\infty = 2\left(\frac{E}{\rho}\right)^{1/2} \frac{H}{a^2}. \tag{3.12'}$$

The solutions (3.5) and (3.6) involve eight arbitrary constants for each value of n. However, for $n = 0$ and $n = 1$ only six of these have physical significance. The quantities $C_{3,0}$ and $C_{3,1}$ may be omitted as they do not enter into expressions for displacements and stresses. Furthermore, in order that the displacements u and v in radial and circumferential direction be single valued we must have (see [4] for the case $n = 0$) the relations

$$C_{2,0} = 0, \qquad C_{2,1} = 0. \tag{3.13}$$

We next write resultants and couples in a form which corresponds to (3.1), viz.,

$$N_r = \begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix} e^{ipt} N_{r,n} , \quad \text{etc.} \tag{3.14}$$

In this way we shall have in terms of the functions F_n and w_n

$$\left.\begin{aligned} N_{r,n} &= \frac{1}{r}\frac{dF_n}{dr} - \frac{n^2}{r^2}F_n\,,\\ N_{\theta,n} &= \frac{d^2F_n}{dr^2}\,, \qquad N_{r\theta,n} = \mp\frac{d}{dr}\left(\frac{n}{r}F_n\right)\end{aligned}\right\} \tag{3.15}$$

$$\left.\begin{aligned} V_{r,n} &= -D\left[\frac{dL_nw_n}{dr} - (1-\nu)\frac{n^2}{r}\frac{d}{dr}\left(\frac{w_n}{r}\right)\right] - \frac{r}{R}N_{r,n}\\ V_{\theta,n} &= -D\left[\mp\frac{n}{r}w_n \mp (1-\nu)n\frac{d^2}{dr^2}\left(\frac{w_n}{r}\right)\right] - \frac{r}{R}N_{r\theta,n}\end{aligned}\right\} \tag{3.16}$$

$$\left.\begin{aligned} M_{r,n} &= -D\left[\frac{d^2w_n}{dr^2} + \frac{\nu}{r}\frac{dw_n}{dr} - \frac{\nu n^2}{r^2}w_n\right]\\ M_{\theta,n} &= -D\left[\frac{1}{r}\frac{dw_n}{dr} - \frac{n^2}{r^2}w_n + \nu\frac{d^2w_n}{dr^2}\right]\\ M_{r\theta,n} &= \pm(1-\nu)Dn\frac{d}{dr}\left(\frac{w_n}{r}\right)\end{aligned}\right\} \tag{3.17}$$

4. Boundary conditions for spherical segment with free edge. In order that the edge $r = a$ be a free edge the following four conditions must be satisfied

$$N_{r,n}(a) = 0, \qquad N_{r\theta,n}(a) = 0, \tag{4.1}$$

$$M_{r,n}(a) = 0, \qquad V_{r,n}(a) = 0. \tag{4.2}$$

In terms of F_n and w_n and indicating differentiation with respect to r by primes, these conditions assume the following form

$$\left.\begin{aligned} aF_n'(a) - n^2F_n(a) &= 0\\ n[aF_n'(a) - F_n(a)] &= 0\end{aligned}\right\}, \tag{4.3}$$

$$\left.\begin{aligned} a^2w_n''(a) + a\nu w_n'(a) - \nu n^2w_n(a) &= 0\\ a^2[L_nw_n(a)]' - n^2(1-\nu)[aw_n'(a) - w_n(a)] &= 0\end{aligned}\right\}. \tag{4.4}$$

Equations (4.3) may be simplified further, to read

$$\left.\begin{aligned} &F_0'(a) = 0, \qquad aF_1'(a) = F_1(a)\\ &F_n'(a) = 0, \qquad F_n(a) = 0; \qquad 2 \le n\end{aligned}\right\}. \tag{4.5}$$

In addition to boundary conditions for $r = a$ we have regularity conditions for $r = 0$. Inspection of (3.5) to (3.11) reveals that these regularity conditions require the vanishing of four of the eight constants of integration in (3.5) and (3.6), namely

$$C_{2,n} = C_{4,n} = A_{2,n} = A_{4,n} = 0. \tag{4.6}$$

This leaves us with the following expressions for w_n and F_n

$$w_n = \frac{1}{RD\lambda^4}C_{1,n}r^n + A_{1,n}J_n(\lambda r) + A_{3,n}I_n(\lambda r), \tag{4.7}$$

$$F_n = \left(1 + \frac{C}{R^2 D\lambda^4}\right) C_{1,n} \frac{r^{n+2}}{4(n+1)} + C_{3,n} r^n + \frac{C}{R\lambda^2} [-A_{1,n} J_n(\lambda r) + A_{3,n} I_n(\lambda r)]. \tag{4.8}$$

An equation determining frequencies of free vibrations is now obtained by substituting the solutions (4.7) and (4.8) in the boundary conditions (4.4) and (4.5). In doing this it is convenient to consider the cases $n = 0, 1$ separately from the general and more complicated case $2 \leq n$.

5. Rotationally symmetric vibrations and vibrations with one nodal diameter. Substitution of (4.7) and (4.8) into (4.4) and (4.5) for $n = 0$ and $n = 1$ leads to the following system of simultaneous homogeneous equations for the constants of integration in (4.7) and (4.8)

$$\left.\begin{aligned} A_{1,0}[\mu J_0(\mu) + (1 - \nu) J_0'(\mu)] - A_{3,0}[\mu I_0(\mu) - (1 - \nu) I_0'(\mu)] &= 0 \\ A_{1,0} J_0'(\mu) - A_{3,0} I_0'(\mu) &= 0 \\ \frac{1}{2}\left[1 + \frac{Ca^4}{R^2 D\mu^4}\right] C_{1,0} + \frac{C}{R\mu} [-A_{1,0} J_0'(\mu) + A_{3,0} I_0'(\mu)] &= 0 \end{aligned}\right\} \tag{5.1}$$

$$\left.\begin{aligned} &A_{1,1}[(1 - \nu)(J_1(\mu) - \mu J_1'(\mu)) - \mu^2 J_1(\mu)] \\ &\qquad + A_{3,1}[(1 - \nu)(I_1(\mu) - \mu I_1'(\mu)) + \mu^2 I_1(\mu)] = 0 \\ &A_{1,1}[(1 - \nu)(-J_1(\mu) + \mu J_1'(\mu)) + \mu^3 J_1'(\mu)] \\ &\qquad + A_{3,1}[(1 - \nu)(-I_1(\mu) + \mu I_1'(\mu)) - \mu^3 I_1'(\mu)] = 0 \\ &\frac{1}{4}\left[1 + \frac{Ca^4}{R^2 D\mu^4}\right] C_{1,1} + \frac{C}{aR\mu^2} [A_{1,1}(J_1(\mu) - \mu J_1'(\mu)) \\ &\qquad + A_{3,1}(I_1(\mu) + \mu I_1'(\mu))] = 0 \end{aligned}\right\}. \tag{5.2}$$

In these equations the quantity μ is defined as

$$\mu = \lambda a = \left[\frac{h\rho a^4}{D} p^2 - \frac{Ca^4}{DR^2}\right]^{1/4} \tag{5.3}$$

or, with the reference frequency p_∞ of Eq. (3.12)

$$\mu = \left[\frac{h\rho a^4}{D} (p^2 - p_\infty^2)\right]^{1/4}. \tag{5.3'}$$

It is noted that the systems (5.1) and (5.2) are such that in each case in order to determine admissible values of μ it is sufficient to consider the first two equations of the system. These first two equations are the same as the corresponding equations for vibrations of flat plates and accordingly the equation determining possible values of μ is the same as Kirchhoff's frequency equation for vibrations of circular plates with free edge, with zero or one nodal diameter,

$$\frac{\mu}{2}\left[\frac{J_n(\mu)}{J_{n+1}(\mu)} + \frac{I_n(\mu)}{I_{n+1}(\mu)}\right] = 1 - \nu, \qquad n = 0, 1. \tag{5.4}$$

We note from (5.3′) that values of p larger and smaller than p_∞ are accounted for by considering the following ranges of μ.

$$p \geq p_\infty : \mu = \zeta e^{im\pi/2}; \qquad \zeta > 0, \qquad m = 0, 1, 2, \cdots \tag{5.5}$$

$$p \leq p_\infty : \mu = \zeta e^{im\pi/4}; \qquad \zeta > 0, \qquad m = 1, 3, 5, \cdots \tag{5.6}$$

However, the characteristic equation (5.4) can be shown to possess no solutions of the form (5.6). Also, the substitution of (5.5) into (5.4) leaves (5.4) unchanged so that it suffices to consider only real positive values of μ. We conclude that for $n = 0, 1$, the frequency p_∞ represents the lowest possible frequency of free vibration.

For given ν Eq. (5.4) is satisfied by an infinite sequence of positive values $\mu = \mu_{n,k}$, $k = 1, 2, 3, \cdots$, where k is the number of nodal circles. Let $p_{n,k}$ be the corresponding frequencies of free vibrations and $p_{n,k}^{(0)}$ the values of these frequencies for the case of flat plates, for which $p_\infty = 0$. According to (5.3′) we have

$$p_{n,k}^{(0)} = \mu_{n,k}^2 \left(\frac{D}{h\rho a^4}\right)^{1/2} \tag{5.7}$$

and

$$p_{n,k}^2 = p_{n,k}^{(0)2} + p_\infty^2 . \tag{5.8}$$

We may write (5.8) in the form

$$\frac{p_{n,k}}{p_{n,k}^{(0)}} = \left[1 + \left(\frac{p_\infty}{p_{n,k}^{(0)}}\right)^2\right]^{1/2} \tag{5.9}$$

or with

$$\left(\frac{p_\infty}{p_{n,k}^{(0)}}\right)^2 = \frac{E}{\rho}\frac{4H^2}{a^4}\frac{h\rho a^4}{D}\frac{1}{\mu_{n,k}^4} = \frac{48(1-\nu^2)}{\mu_{n,k}^4}\left(\frac{H}{h}\right)^2,$$
$$\frac{p_{n,k}}{p_{n,k}^{(0)}} = \left[1 + \frac{48(1-\nu^2)}{\mu_{n,k}^4}\right]^{1/2}. \tag{5.10}$$

Known values of $\mu_{n,k}$, for $\nu = 1/3$ and for $k = 1, 2, 3$, are given in Table 1. Values of the frequency ratio $p_{n,k}/p_{n,k}^{(0)}$ as a function of H/h may be found in Figs. 2 and 3. These results are new only for $n = 1$, the case $n = 0$ having previously been considered in [2]. We note that the modes with no nodal circles ($k = 0$) correspond to rigid-body translations and rotations.

We finally note that the frequency $p_{n,k}$ is equal to p_∞ in the limiting case $D = 0$, or $H/h = \infty$. In this sense, the frequency p_∞ may be designated as the membrane vibration frequency of the shell.

6. Vibrations with two or more nodal radii. Introduction of the solutions (4.7) and (4.8) into the boundary conditions (4.4) and (4.5) leads, when $2 \leq n$, to the following system of simultaneous equations.

$$\left(1 + \frac{\kappa^4}{\mu^4}\right)\frac{1}{4(n+1)} C_{1,n}^* + C_{3,n}^* + \frac{1}{\mu^2}\left[-A_{1,n}J_n(\mu) + A_{3,n}I_n(\mu)\right] = 0, \tag{6.1}$$

$$\left(1 + \frac{\kappa^4}{\mu^4}\right)\frac{n+2}{4(n+1)} C_{1,n}^* + nC_{3,n}^* + \frac{1}{\mu}\left[-A_{1,n}J_n'(\mu) + A_{3,n}I_n'(\mu)\right] = 0, \tag{6.2}$$

$$\begin{aligned}(1-\nu)n(n-1)\frac{\kappa^4}{\mu^4} C_{1,n}^* &+ \{[(1-\nu)n^2 - \mu^2]J_n(\mu) - (1-\nu)\mu J_n'(\mu)\}A_{1,n} \\ &+ \{[(1-\nu)n^2 + \mu^2]I_n(\mu) - (1-\nu)\mu I_n'(\mu)\}A_{3,n} = 0,\end{aligned} \tag{6.3}$$

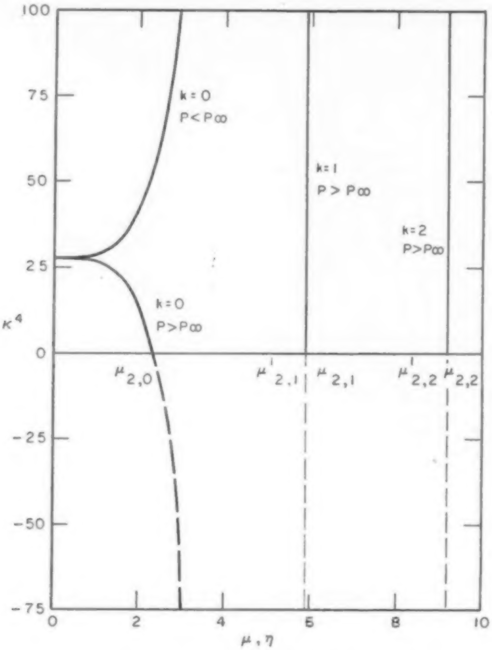

FIG. 1. Curves $\mu = \mu(\kappa^4)$ and $\eta = \eta(\kappa^4)$ when $\nu = 1/3$, according to Eqs. (6.7) and (6.13).

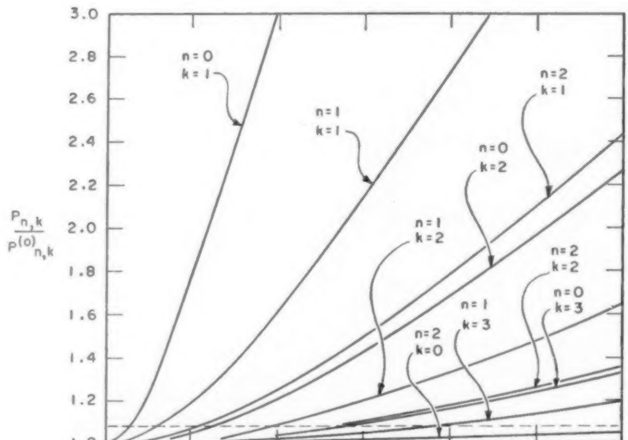

FIG. 2. Values of ratio of shell frequency $p_{n,k}$ to plate frequency $p^{(0)}_{n,k}$ in dependence on ratio of shell rise H to shell thickness h for various values of the number $2n$ of nodal radii and the number k of nodal circles, for a value $\nu = 1/3$ of Poisson's ratio.

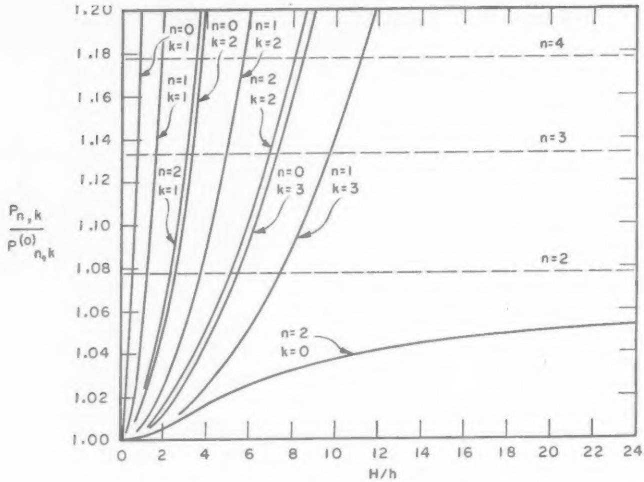

FIG. 3. Enlarged version of part of Fig. 2.

$$(1 - \nu)n^2(1 - n)\frac{\kappa^4}{\mu^4}C^*_{1,n} + \{(1 - \nu)n^2J_n(\mu) - [(1 - \nu)n^2\mu + \mu^3]J'_n(\mu)\}A_{1,n}$$
$$+ \{(1 - \nu)n^2I_n(\mu) - [(1 - \nu)n^2\mu - \mu^3]I'_n(\mu)\}A_{3,n} = 0. \tag{6.4}$$

The quantities $C^*_{m,n}$ and κ are defined as follows.

$$C_{1,n} = \frac{C}{Ra^n}C^*_{1,n}, \qquad C_{3,n} = \frac{C}{Ra^{n-2}}C^*_{3,n} \tag{6.5}$$

and

$$\kappa^4 = \frac{Ca^4}{DR^2} = 48(1 - \nu^2)\left(\frac{H}{h}\right)^2. \tag{6.6}$$

The vanishing of the determinant of the system (6.1) to (6.4) leads to the following frequency equation

$$p_\infty \leq p : \frac{\mu^4}{\kappa^4} = \frac{S_n(\mu)}{R_n(\mu)} - 1, \tag{6.7}$$

where

$$S_n(\mu) = 4n^2(n^2 - 1)(1 - \nu)\Big\{\mu[J_n(\mu)I'_n(\mu) - J'_n(\mu)I_n(\mu)]$$
$$+ (n + 1)(1 - \nu)\left[I'_n(\mu) - \frac{n}{\mu}I_n(\mu)\right]\left[J'_n(\mu) - \frac{n}{\mu}J_n(\mu)\right]\Big\} \tag{6.8}$$

and

$$R_n(\mu) = \{(1 - \nu)[\mu J'_n(\mu) - n^2J_n(\mu)] + \mu^2J_n(\mu)\}\{(1 - \nu)n^2[\mu I'_n(\mu) - I_n(\mu)] - \mu^3I'_n(\mu)\}$$
$$- \{(1 - \nu)n^2[\mu J'_n(\mu) - J_n(\mu)] + \mu^3J'_n(\mu)\}\{(1 - \nu)[\mu I'_n(\mu) - n^2I_n(\mu)] - \mu^2I_n(\mu)\}. \tag{6.9}$$

In order to account for values of p both larger and smaller than p_∞ we consider the ranges of μ given in (5.5) and (5.6). Equation (6.7) is unchanged by the substitution (5.5) so that it will suffice to consider real positive values of μ for the range $p \geq p_\infty$.

$$p \geq p_\infty ; \qquad \mu = \left[\frac{h\rho a^4}{D}(p^2 - p_\infty^2)\right]^{1/4} \geq 0. \tag{6.10}$$

In contrast to the cases $n = 0, 1$, Eq. (6.7) does possess solutions of the form (5.6), implying that values of $p < p_\infty$ are possible for $n \geq 2$. Equation (6.7) remains the same for all μ of form (5.6) so that it is sufficient to consider the substitution

$$p \leq p_\infty ; \qquad \mu = i^{3/2}\eta, \qquad \eta \geq 0. \tag{6.11}$$

Introducing Kelvin functions of order n by the relations

$$\left.\begin{aligned} J_n(i^{3/2}\eta) &= \mathrm{ber}_n\eta + i\,\mathrm{bei}_n\eta \\ I_n(i^{3/2}\eta) &= \exp\left(-\frac{n\pi i}{2}\right)(\mathrm{ber}_n\eta - i\,\mathrm{bei}_n\eta) \end{aligned}\right\} \tag{6.12}$$

we write Eq. (6.7) in the form

$$p \leq p_\infty ; \qquad \frac{\eta^4}{\kappa^4} = \frac{U_n(\eta)}{T_n(\eta)} + 1, \tag{6.13}$$

where

$$\begin{aligned} U_n(\eta) = 2(1-\nu)n(n^2-1)\Big\{&2^{3/2}n\eta(\mathrm{ber}_n'\eta\,\mathrm{bei}_n\eta - \mathrm{ber}_n\eta\,\mathrm{bei}_n'\eta) \\ &+ 2^{1/2}(1-\nu)n(n+1)\left[\frac{n^2}{\eta^2}(\mathrm{ber}_n^2\eta + \mathrm{bei}_n^2\eta)\right. \\ &\left.-\frac{n}{\eta}(\mathrm{ber}_n^2\eta + \mathrm{bei}_n^2\eta)' + (\mathrm{ber}_n'\eta)^2 + (\mathrm{bei}_n'\eta)^2\right]\Big\} \end{aligned} \tag{6.14}$$

and

$$\begin{aligned} T_n(\eta) = {}& [(1-\nu)^2n^2(n^2-1) - \eta^4]2^{1/2}\eta(\mathrm{ber}_n'\eta\,\mathrm{bei}_n\eta - \mathrm{ber}_n\eta\,\mathrm{bei}_n'\eta) \\ &+ 2^{1/2}(1-\nu)\eta^4\left\{n^2\left[\frac{1}{\eta}(\mathrm{ber}_n^2\eta + \mathrm{bei}_n^2\eta)\right]' - (\mathrm{ber}_n'\eta)^2 - (\mathrm{bei}_n'\eta)^2\right\}. \end{aligned} \tag{6.15}$$

Equations (6.7) and (6.13) assume the same limiting form when $p = p_\infty$, that is when μ and η, respectively, are zero, namely

$$\begin{aligned} p = p_\infty ; \qquad \kappa^4 = (1-\nu)(3+\nu)n^2(n^2-1)\Big[&1 + \tfrac{1}{4}(1-\nu)(n-2) \\ &- \frac{n^2(n-1)(1-\nu)(4n+9-\nu)}{16(n+2)^2(n+3)}\Big]^{-1}. \end{aligned} \tag{6.16}$$

7. Solution of frequency equations for $n = 2$. As the character of the frequency calculations is the same for all cases $n \geq 2$ we limit ourselves here to the case $n = 2$. According to (6.10) and (6.11) we have that the frequency p is given in the form

$$p \geq p_\infty ; \qquad p^2 = p_\infty^2 + \frac{D}{\rho h a^4}\mu^4, \tag{7.1}$$

$$p \leq p_\infty ; \qquad p^2 = p_\infty^2 - \frac{D}{\rho h a^4} \eta^4. \tag{7.2}$$

The quantities μ and η are determined as functions of $\kappa^4 = 48(1 - \nu^2)(H/h)^2$ by solving Eqs. (6.7) and (6.13). We carry out this solution by finding κ^4 as a function of μ and η.

Considering first Eq. (6.7) we find that κ^4 has a value slightly above 25 when $\mu = 0$ which decreases to zero when $\mu = \mu_{2,0} = 2.30$. For larger values of μ, from $\mu = \mu_{2,0}$ to $\mu = \mu'_{2,1}$ we have κ^4 negative which means that this part of the curve has no physical significance. From $\mu = \mu'_{2,1}$ to $\mu = \mu_{2,1}$ the values of κ^4 are again positive. The next branch of the curve for positive κ^4 lies between the values $\mu'_{2,2}$ and $\mu_{2,2}$. The pattern repeats itself with increasing values of μ (see Fig. 1).

We note that the values $\mu_{2,0}$, $\mu_{2,1}$, $\mu_{2,2}$, etc., are roots of Eq. (6.7) for $\kappa^4 = 0$, that is roots of the corresponding flat-plate frequency equation

$$R_2(\mu) = 0. \tag{7.3}$$

The values $\mu'_{2,1}$, $\mu'_{2,2}$, etc., are roots of (6.7) for $\kappa^4 = \infty$, that is roots of the equation

$$S_2(\mu) - R_2(\mu) = 0. \tag{7.4}$$

We note that for $k = 1, 2, \cdots$ the values of $\mu_{2,k}$ and of $\mu'_{2,k}$ are so nearly the same that for the purpose of frequency calculations we may disregard their difference. This means that for $n = 2$ and $k \geq 1$ (and by implication for $n > 2$) we may take for practical purposes μ from the flat-plate frequency equation, just as for the cases $n = 0$ and $n = 1$. However, in addition to the regime in which this is possible we now have the regime $k = 0$ where this is not possible. Numerical values of $\mu_{2,k}^2$ are given in Table 1.

We now turn to Eq. (6.13) and determine η as a function of κ^4. We find a curve which gives κ^4 somewhat greater than 25 for $\eta = 0$, with increasing values of κ^4 as η increases. It is apparent that the $\eta(\kappa^4)$-curve represents a continuation of the $\mu(\kappa^4)$-curve for $k = 0$. The common point of these two branches for $\mu = \eta = 0$ is, as it should be, given by Eq. (6.16).

On the basis of the data contained in Fig. 1 we have calculated the frequency curves $n = 2$, $k = 0, 1, 2$ as contained in Figs. 2 and 3. It is apparent that there is an essential difference between the case $k = 0$ where we have no nodal circles and the cases $k \geq 1$ where we have one or more nodal circles. For $k = 0$ the influence of shell curvature remains insignificant, the frequency being very nearly equal to the corresponding flat-plate frequency. For $k \geq 1$ the influence of shell curvature is qualitatively significant.

TABLE 1. $\mu_{n,k}^2$ *for* $\nu = 1/3$.

n \ k	0	1	2	3
0	—	9.076	38.52	87.82
1	—	20.52	59.86	119.0
2	5.251	35.24	84.38	153.34
3	12.23	52.91	112.0	190.70
4	21.49	73.9	142.5	231.03

8. Frequencies of membrane vibrations. The differential equations of membrane theory follow from (2.1) to (2.5) by setting in them

$$D = 0. \tag{8.1}$$

If this is done the differential equations (2.1) and (2.2) may be reduced to the following form

$$\nabla^2\left[\rho h \frac{\partial^2 w}{\partial t^2} + \frac{C}{R^2} w\right] \doteq 0, \tag{8.2}$$

$$\frac{1}{R} \nabla^2 F = -\rho h \frac{\partial^2 w}{\partial t^2}. \tag{8.3}$$

Equation (8.2), with $w = \begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix} w_n(r)e^{ipt}$, is satisfied either when

$$p^2 = \frac{C}{R^2 h\rho} = p_\infty^2 \tag{8.4}$$

or, when

$$w_n = A_n r^n + B_n r^{-n}, \qquad 1 \leq n, \tag{8.5}$$

$$w_0 = A_0 + B_0 \log r. \tag{8.6}$$

In either case F_n is found in terms of w_n by integrating the relation

$$L_n F_n = \rho h R p^2 w_n\,. \tag{8.7}$$

Since $D = 0$ only the first two of the four boundary conditions (4.1) and (4.2) need to be considered. Appropriate substitution of (8.5) to (8.7) shows that these equations are satisfied only when $p = 0$.

There remains the frequency $p = p_\infty$ which is associated with arbitrary functions w_n, insofar as the differential equation (8.2) is concerned. We have already seen in Sec. 5 that the frequency p_∞ is the limiting solution for all modes $n = 0, 1$, and $k \geq 1$, as $D \to 0$, or equivalently as $H/h \to \infty$. For $n \geq 2$ inspection of the solutions (4.7) and (4.8) and of Eqs. (6.1) to (6.4) reveals that the conditions

$$D \to 0, \qquad \kappa^4 \to \infty \tag{8.8}$$

are equivalent. Furthermore, for any given mode $k \geq 1$, μ as given by (6.7) remains finite as $\kappa^4 \to \infty$. Equation (7.1) indicates, therefore, that for modes with one or more nodal circles,

$$\lim_{D\to 0} p \to p_\infty\,, \qquad k \geq 1. \tag{8.9}$$

However, since the one root of (6.13) tends to infinity at the same time that $\kappa^4 \to \infty$ we have that the frequency $p_{n,0}$ given by (7.2) associated with no nodal circles does not tend to the frequency p_∞. In the above sense we consider the frequency p_∞ as the mem-

brane vibration frequency of the shallow spherical shell. Figure 4 which represents values of $p_\infty/p_{n,k}$ as functions of H/h illustrates the way in which the membrane frequency is approached.

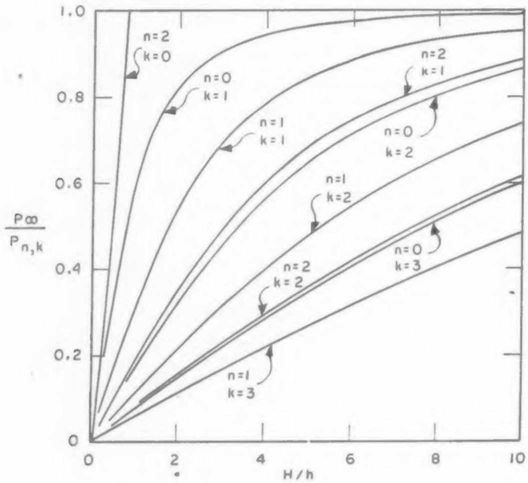

FIG. 4. Relation between shell frequency $p_{n,k}$ and membrane frequency p_∞, when $\nu = 1/3$.

9. Frequencies of inextensional vibrations. The differential equations for inextensional vibrations are obtained from (2.1) and (2.2) by setting in these equations

$$\frac{1}{C} = 0. \tag{9.1}$$

It has been shown earlier [3] that inextensional vibrations of shallow spherical shells with free edge can occur only for the modes $k = 0$, $n \geq 2$, with frequency p_n given by

$$\frac{H}{h} = 0; \qquad p_n = \left(\frac{D}{\rho h a^4}\right)^{1/2} \mu_{n,0}^2 , \tag{9.2}$$

$$\frac{H}{h} > 0; \qquad p_n = \left(\frac{D}{\rho h a^4}\right)^{1/2} [4(1 - \nu)(n^2 - 1)n^2]^{1/2}. \tag{9.3}$$

In order to see to what extent these results are contained as limiting cases in our present calculation we note that $1/C = 0$ is equivalent, just as $D = 0$ is, to

$$\kappa^4 = \infty \tag{9.4}$$

provided H is different from zero.

Since when $\kappa^4 \to \infty$ we have $p \to p_\infty$ for $k = 1, 2, \cdots$ it remains to consider what happens when $k = 0$. According to Fig. 1 the frequency equation in this case for sufficiently large values of κ^4 is Eq. (6.13). As κ^4 tends to infinity η also tends to infinity. We find by means of the asymptotic expansions for Kelvin functions that (6.13) assumes

the following limiting form for large η,

$$\kappa^4 \simeq \eta^4 + 4(1 - \nu)(n^2 - 1)n^2. \tag{9.5}$$

Substitution of (9.5) into (7.2) reduces (7.2) to the following form

$$p^2 \simeq \frac{D}{\rho h a^4} 4(1 - \nu)(n^2 - 1)n^2 \tag{9.6}$$

which shows that when $k = 0$, $H \neq 0$ then when $h \to 0$ we have $p \to p_n$ as given by (9.3).

On the other hand, for $k = 0$, $h \neq 0$, letting $H \to 0$ in (7.1) results in $p \to p_n$ as given by (9.2).

The foregoing considerations indicate that our frequency formula, for $k = 0$, contains as limiting cases both the appropriate flat-plate frequency formula and the frequency formula for inextensional vibrations of shallow shells. The way in which the transition from one of these formulas to the other takes place is clearly seen by means of the curve which in Figs. 2 and 3 is labeled $n = 2$, $k = 0$.

10. The energy of stretching and the energy of bending. Qualitatively, in vibrations which are approximately membrane vibrations the strain energy of stretching will be large compared with the strain energy of bending. On the other hand, for vibrations which are approximately inextensional the strain energy of bending will be large compared with the strain energy of stretching.

We consider in this section the quantitative aspects of the foregoing statement by calculating the ratio

$$\frac{V_s}{V_b + V_s},$$

where V_s is the strain energy of stretching and V_b is the strain energy of bending. For shallow shells these quantities are given by the following expressions:

$$V_s = \int_0^a \int_0^{2\pi} \frac{1}{2C} \left\{ (\nabla^2 F)^2 + 2(1 + \nu)\left[\left(\frac{\partial}{\partial r}\left(\frac{1}{r} \frac{\partial F}{\partial \theta} \right) \right)^2 - \frac{\partial^2 F}{\partial r^2}\left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} \right) \right] \right\} r\, d\theta\, dr, \tag{10.1}$$

$$V_b = \int_0^a \int_0^{2\pi} \frac{D}{2} \left\{ (\nabla^2 w)^2 + 2(1 - \nu)\left[\left(\frac{\partial}{\partial r}\left(\frac{1}{r} \frac{\partial w}{\partial \theta} \right) \right)^2 - \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} \right) \right] \right\} r\, d\theta\, dr. \tag{10.2}$$

We omit the details of the lengthy calculations involved in the evaluation of (10.1) and (10.2). The results of these calculations are incorporated in Fig. 5 which shows how, for $k \geq 1$, the transition from plate to shell is associated with a transfer of strain energy from bending to stretching and how, for $k = 0$, we start with bending energy, encounter a certain amount of stretching energy for moderately curved shells, which, with increasing curvature, goes over into bending energy.

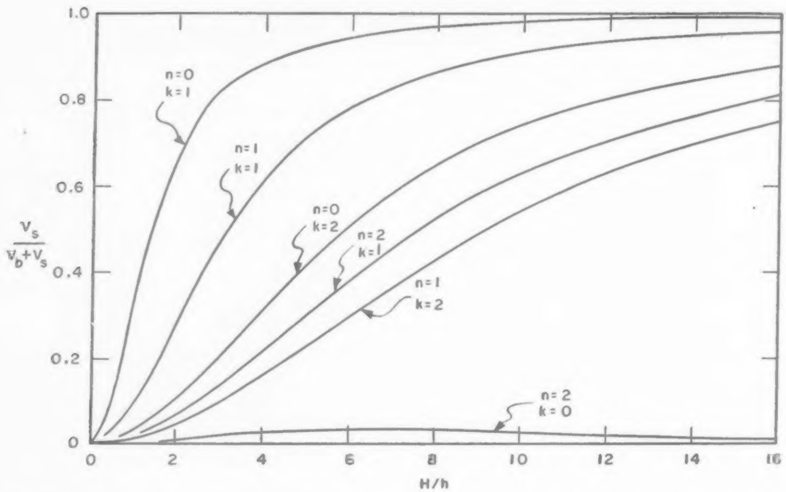

FIG. 5. Ratio of stretching component of strain energy to sum of stretching component and bending component as function of n, k and H/h, when $\nu = 1/3$.

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THERMO-ELASTIC SIMILARITY LAWS*

BY

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1. Introduction. In a book edited by Hetényi [1], Mindlin and Salvadori have discussed certain similarity laws for thermo-elastic problems. They were, however, primarily concerned with the replacement of thermo-elastic problems by purely elastic problems involving dislocations. In the present paper we are concerned with the conditions which must be satisfied by the physical parameters describing the properties of a body and a scale model (scaled both in linear dimensions and time) in order that the stress in the original body may be determined from that at the corresponding position and time in the scale model by means of the multiplier M (= rigidity modulus of body/rigidity modulus of model). It will be seen that the conditions for the stresses in the body to be an arbitrary constant C times those in the model are then easily obtained. In deriving these conditions much of the mathematical analysis of Mindlin and Salvadori could have been used. However, we have preferred to obtain them by somewhat different methods.

We consider that surface tractions and temperatures are specified on the surface of a body and a scale model, the surface tractions and temperatures on the body being respectively M and Θ_0 (a constant) times those at corresponding points and times in the scale model. The dimensions of the body are assumed to be l times those of the model and any time for the body is τ_0 times the corresponding time for the model.

It is then found that for three-dimensional thermo-elastic problems, the stresses in the original body are M times those in the model provided that

(i) Poisson's ratio σ is the same for the model and the original body;

(ii) β $(= \Theta_0\nu/\mu)$ is the same for the model and original body, where ν is the coefficient of thermal expansion, μ is the rigidity modulus and $\Theta_0 = 1$ for the model; and

(iii) α $(= \tau_0\kappa/l^2)$ is the same for the model and original body, where κ is the thermal diffusivity and $\tau_0 = 1$, $l = 1$ for the model.

If the body is subjected to plane thermo-elastic strain, the stresses in the body are M times those in the model provided that β, σ and α have the same values for the model and the original body. If the surface tractions applied to each closed boundary of the body considered are self-equilibrating, the stresses in the body are M times those in the scale model provided that $\beta(1 - 2\sigma)/(1 - \sigma)$ and α have the same values for the model and the original body.

In a generalized plane stress problem, the average stresses are M times those in the model provided that β, σ and α have the same values for the body and the model. If the surface tractions applied to each closed boundary of the body are self-equilibrating, the conditions become that α and $\beta(1 - 2\sigma)$ have the same values in the model and the original body.

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Finally, we have the case of a body in plane thermo-elastic strain and a scale model in generalized plane thermo-elastic stress, the surface temperatures in the body being Θ_0 times the average surface temperatures in the model, while the surface tractions in the body are M times the average surface tractions in the model, the averages being taken over the thickness of the generalized plane stress model. Then, the stress in the body is M times the average stress at the corresponding position and time in the model, provided that σ, β and α have the same values in the model and the original body. However, if the surface tractions applied to each closed boundary of the body are self-equilibrating, these conditions can be replaced by the weaker conditions

$$\alpha^* = \alpha \text{ and } \beta^*(1 - 2\sigma^*) = \beta(1 - 2\sigma)/(1 - \sigma),$$

where σ^*, β^* and α^* are the values of σ, β and α respectively for the model.

In all the cases considered, it is assumed that no body forces are applied and that, although the surface tractions and temperatures may vary with time, such changes are sufficiently slow for inertial forces to be neglected. In the plane problems, the similarity laws apply, of course, only to the planar stress components.

In each of the cases discussed, the modelling conditions given above are those for which the stresses in the body considered are M times those in the model. Since the governing equations of the problems considered are linear, we can readily derive conditions under which the stresses in the original body shall be any constant C times those in the model. For example, let us consider the case in which the original body is in plane strain and the model is in generalized plane stress, and the surface tractions applied to each closed boundary of the body are self-equilibrating. The condition $\beta^*(1 - 2\sigma^*) = \beta(1 - 2\sigma)/(1 - \sigma)$ implies that

$$\Theta_0 = \frac{M}{N} \frac{(1 - 2\sigma^*)(1 - \sigma)}{1 - 2\sigma},$$

where N is the ratio of the thermal expansion coefficients for the original body and the model. If we now multiply both the surface tractions and temperatures in the scale model by M/C the stresses in the original body will be C times the average stresses in the model and Θ_0 will be changed to

$$\Theta_0 = \frac{C}{N} \frac{(1 - 2\sigma^*)(1 - \sigma)}{1 - 2\sigma}.$$

Similar results may readily be obtained in the other cases considered.

2. Fundamental equations. We use Cartesian tensor notation and denote a rectangular coordinate system by y_i, where Latin indices take the values 1, 2, 3. Components of the displacement, strain, stress and surface traction (or stress vector) are denoted by v_i, ϵ_{ij}, τ_{ij} and F_i respectively and τ is time and θ temperature. If motion in the elastic body is such that inertial terms may be neglected, and if body forces are zero, then

$$\frac{\partial \tau_{ij}}{\partial y_i} = 0. \tag{2.1}$$

The stress-strain relations [2] are

$$\tau_{ij} = 2\mu\left(\epsilon_{ij} + \frac{\sigma}{1 - 2\sigma} \epsilon_{rr}\delta_{ij}\right) - \nu\theta\delta_{ij}$$

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial y_j} + \frac{\partial v_j}{\partial y_i}\right), \tag{2.2}$$

and the equation of heat conduction is

$$\frac{\partial^2 \theta}{\partial y_i \, \partial y_i} = \frac{1}{\kappa}\frac{\partial \theta}{\partial t}. \tag{2.3}$$

In (2.2) δ_{ij} is a Kronecker delta, μ and σ are the rigidity modulus and Poisson's ratio respectively, and ν is a coefficient of thermal expansion. In (2.3) κ is the thermal diffusivity.

We now express (2.1), (2.2) and (2.3) in non-dimensional form by using the substitutions

$$\begin{aligned} y_i &= lx_i\,, & v_i &= lu_i\,, & \tau &= \tau_0 t, & \epsilon_{ij} &= e_{ij}\,, \\ && \tau_{ij} &= \mu t_{ij}\,, & F_i &= \mu f_i\,, & \theta &= \theta_0 T, \end{aligned} \tag{2.4}$$

where l is a standard length, τ_0 a standard time and Θ_0 a standard temperature. Also, x_i, u_i, t, e_{ij}, t_{ij}, f_i and T are non-dimensional coordinates, displacements, time, strains, stresses, surface tractions (or stress vectors) and temperature respectively. Using (2.4), Eqs. (2.1), (2.2) and (2.3) become

$$\frac{\partial t_{ij}}{\partial x_j} = 0, \tag{2.5}$$

$$\begin{aligned} t_{ij} &= 2\left(e_{ij} + \frac{\sigma}{1 - 2\sigma} e_{rr}\delta_{ij}\right) - \beta T \delta_{ij}\,, \\ e_{ij} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \end{aligned} \tag{2.6}$$

and

$$\frac{\partial^2 T}{\partial x_i \, \partial x_i} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2.7}$$

respectively, where

$$\alpha = \frac{\tau_0 \kappa}{l^2} \quad \text{and} \quad \beta = \frac{\Theta_0 \nu}{\mu}. \tag{2.8}$$

The stress components t_{ij} may be eliminated from (2.6) and (2.7) to give the three differential equations

$$\frac{\partial^2 u_i}{\partial x_j \, \partial x_j} + \frac{1}{1 - 2\sigma}\frac{\partial^2 u_j}{\partial x_i \, \partial x_j} - \beta \frac{\partial T}{\partial x_i} = 0. \tag{2.9}$$

In addition to the above fundamental equations, we have boundary conditions in which temperature or temperature gradients, displacements or surface tractions—or a mixture of such conditions—are prescribed. For clarity, we restrict our attention to boundary conditions which involve no new parameters, although it would not be difficult to include these.

If F_i is the surface traction per unit area acting on a surface, the normal to which

has direction-cosines l_i , then

$$F_i = \tau_{ij} l_j . \tag{2.10}$$

We see that the non-dimensional surface traction f_i is given by

$$f_i = t_{ij} l_j . \tag{2.11}$$

3. Similarity laws in three dimensions. If temperature or temperature gradients are prescribed on the boundaries of the body considered, then we see from (2.7) that the non-dimensional temperature distribution throughout the body depends only on the non-dimensional coordinates x_i , non-dimensional time t, and the non-dimensional parameter α. If we regard this non-dimensional temperature distribution as being the actual temperature distribution in a scale model of the body under consideration, then the temperature distribution in the original body may be obtained by using the scale relations (2.4), provided α is kept constant. If, however, we are concerned only with a steady state temperature distribution, we see from (2.7) that T is then independent of α, so that no restriction on α is necessary. Further, if the boundary conditions on the temperature do not involve a time scale (e.g. if their dependence on time has the form of a step-function), then we can choose the scale-factor τ_0 in such a way that α is kept constant.

If boundary conditions for the body under consideration are given in terms of displacements, then (2.9) shows that the non-dimensional displacement u_i throughout the body depends, in general, on σ and β. Again, if the boundary conditions are given in terms of surface tractions (i.e. in terms of stress components), we see, from (2.5), (2.6) and (2.11) that the non-dimensional displacement components u_i and stress components t_{ij} throughout the body depend, in general, on σ and β. Thus, if we regard these non-dimensional displacement and stress distributions as being the actual displacement and stress distributions in a scale model of the body under consideration, then the displacement and stress distributions in the original body can, in general, be calculated from those in the model, by means of Eqs. (2.4), only if σ and β have the same values in the model and the original body.

4. Plane strain. The assumptions of plane strain are that

$$u_1 = u_1(x_1 , x_2 , t), \qquad u_2 = u_2(x_1 , x_2 , t), \qquad T = T(x_1 , x_2 , t), \qquad u_3 = 0. \tag{4.1}$$

It follows from (2.5), (2.6) and (2.7) that

$$e_{31} = e_{23} = e_{33} = t_{31} = t_{23} = 0, \tag{4.2}$$

and

$$\frac{\partial t_{\lambda\mu}}{\partial x_\lambda} = 0, \tag{4.3}$$

$$\left.\begin{aligned} t_{\lambda\mu} &= 2\left(e_{\lambda\mu} + \frac{\sigma}{1-2\sigma} e_{\rho\rho}\delta_{\lambda\mu}\right) - \beta T \delta_{\lambda\mu} , \\ e_{\lambda\mu} &= \frac{1}{2}\left(\frac{\partial u_\lambda}{\partial x_\mu} + \frac{\partial u_\mu}{\partial x_\lambda}\right), \end{aligned}\right\} \tag{4.4}$$

$$\frac{\partial^2 T}{\partial x_\lambda \, \partial x_\lambda} = \frac{1}{\alpha}\frac{\partial T}{\partial t} , \tag{4.5}$$

where Greek subscripts take values 1, 2. With the help of complex variable notations, Eqs. (4.3) and (4.4) can be integrated in terms of complex potentials (see [3] and [4]). We use the notation

$$z = x_1 + ix_2 , \qquad D = u_1 + iu_2 , \qquad \Theta = t_{11} + t_{22} , \qquad \Phi = t_{11} - t_{22} + 2it_{12} , \tag{4.6}$$

so that Eqs. (4.3) and (4.4) are replaced by

$$\frac{\partial \Phi}{\partial z} + \frac{\partial \Theta}{\partial \bar{z}} = 0, \tag{4.7}$$

$$\Theta = \frac{2}{1 - 2\sigma}\left(\frac{\partial D}{\partial z} + \frac{\partial \bar{D}}{\partial \bar{z}}\right) - 2\beta T, \qquad \Phi = 4\,\frac{\partial D}{\partial \bar{z}}, \tag{4.8}$$

where a bar placed over a quantity denotes the complex conjugate of that quantity.

From (4.7), it can be shown that Θ and Φ must be expressible in the form

$$\Theta = 4\,\frac{\partial^2 \varphi}{\partial z\,\partial \bar{z}}, \qquad \Phi = -4\,\frac{\partial^2 \varphi}{\partial \bar{z}\,\partial \bar{z}}, \tag{4.9}$$

where φ is a real function (Airy's stress function). The second equations in (4.8) and (4.9) can be integrated to give

$$D + \frac{\partial \varphi}{\partial \bar{z}} = 4(1 - \sigma)f(z), \tag{4.10}$$

where $f(z)$ is an arbitrary function* of z. Using (4.10) and its complex conjugate we may eliminate D and Θ from the first equations in (4.8) and (4.9) to obtain

$$\frac{\partial^2 \varphi}{\partial z\,\partial \bar{z}} = f'(z) + \bar{f}'(\bar{z}) - \frac{\beta(1 - 2\sigma)}{4(1 - \sigma)}\,T. \tag{4.11}$$

Since φ is a real function, two integrations of (4.11) then give

$$\varphi = z\bar{f}(\bar{z}) + \bar{z}f(z) + g(z) + \bar{g}(\bar{z}) - \frac{\beta(1 - 2\sigma)}{1 - \sigma}\,R, \tag{4.12}$$

where $g(z)$ is a second arbitrary function of z and where $R(z, \bar{z}, t)$ is a real particular integral of the equation

$$4\,\frac{\partial^2 R}{\partial z\,\partial \bar{z}} = T. \tag{4.13}$$

From (4.9), (4.10) and (4.12), we now obtain stress and displacement components in the form

$$\begin{aligned} \Theta &= 4[f'(z) + \bar{f}'(\bar{z})] - \frac{\beta(1 - 2\sigma)}{1 - \sigma}\,T, \\ \Phi &= -4[z\bar{f}''(\bar{z}) + \bar{g}''(\bar{z})] + \frac{4\beta(1 - 2\sigma)}{1 - \sigma}\,\frac{\partial^2 R}{\partial \bar{z}\,\partial \bar{z}}, \\ D &= (3 - 4\sigma)f(z) - z\bar{f}'(\bar{z}) - \bar{g}'(\bar{z}) + \frac{\beta(1 - 2\sigma)}{1 - \sigma}\,\frac{\partial R}{\partial \bar{z}}. \end{aligned} \tag{4.14}$$

*$f(z)$ and $g(z)$, defined in (4.12), also depend on t, but this is not shown explicitly.

If X_α are the components of non-dimensional resultant force (per unit length of x_3 axis) due to the stresses acting across an arc AB in the material, in a certain sense, then

$$\begin{aligned} X_1 &= \int_A^B f_1\, ds = \int_A^B (t_{12}\, dx_1 - t_{11}\, dx_2) \\ X_2 &= \int_A^B f_2\, ds = \int_A^B (t_{22}\, dx_1 - t_{12}\, dx_2), \end{aligned} \tag{4.15}$$

where ds is an element of length of the arc AB. Hence, using (4.6),

$$X_1 + iX_2 = \tfrac{1}{2}i \int_A^B (\Theta\, dz - \Phi\, d\bar{z}), \tag{4.16}$$

and with (4.9), this becomes

$$X_1 + iX_2 = 2i\left[\frac{\partial \varphi}{\partial \bar{z}}\right]_A^B, \tag{4.17}$$

where $[\ \ ]_A^B$ denotes the change of value of the argument as we move along the arc from A to B. From (4.17) and (4.12), we have

$$X_1 + iX_2 = 2i\left[f(z) + z\bar{f}'(\bar{z}) + \bar{g}'(\bar{z}) - \frac{\beta(1 - 2\sigma)}{1 - \sigma}\frac{\partial R}{\partial \bar{z}}\right]_A^B. \tag{4.18}$$

5. Single-valued stresses and displacements. We now restrict attention to problems in which the stress and displacement components and the temperature are single-valued and without singularities at every internal point of the body. We also assume at present that the real particular integral R of (4.13) can be chosen to be single-valued, together with its derivatives up to and including the fourth order. This assumption will be justified in the Appendix (Sec. 9). It follows from (4.14) that the contributions of R to the stress and displacement components are single-valued. The conditions for the stress and displacement components to be single-valued therefore reduce to

$$[f'(z) + \bar{f}'(\bar{z})]_A^A = 0, \qquad [z\bar{f}''(\bar{z}) + \bar{g}''(\bar{z})]_A^A = 0$$

and

$$[(3 - 4\sigma)f(z) - z\bar{f}'(\bar{z}) - \bar{g}'(\bar{z})]_A^A = 0, \tag{5.1}$$

where $[\ \ ]_A^A$ denotes the change in value of the function inside the brackets on passing once around a contour in the conventional sense which keeps the area enclosed on the left, the contour lying inside the body.

We may differentiate the last of Eqs. (5.1) with respect to z(or $\bar{z}$), inside the bracket, since we have already assumed that the relevant derivatives exist (see [5]), to obtain

$$[(3 - 4\sigma)f'(z) - \bar{f}'(\bar{z})]_A^A = 0. \tag{5.2}$$

Using this, together with (5.1), we obtain

$$\begin{gathered} [f'(z)]_A^A = 0, \qquad [g''(z)]_A^A = 0, \\ (3 - 4\sigma)[f(z)]_A^A = [\bar{g}'(\bar{z})]_A^A, \end{gathered} \tag{5.3}$$

as given, e.g. by [2] or [3], for the case when the temperature is constant throughout the body.

If we further restrict attention to problems in which the stress system is such that the resultant force acting on any closed curve in the body (or its boundary, if singular points on the boundary are excluded by small indentations of the contour) is zero, then, from (4.18) we see that the conditions (5.3) may be replaced by

$$[f(z)]_A^A = 0, \qquad [g'(z)]_A^A = 0. \tag{5.4}$$

6. Similarity laws for plane strain. Suppose that the body under consideration is in a state of plane thermo-elastic strain and that the region S occupied by the cross-section of the body is finite and multiply-connected and bounded by one or more smooth non-intersecting contours L_0, L_1, $\cdots$, L_n, of which L_0 contains all the others.

In view of (5.4), we shall assume that, for the particular initial and boundary conditions of the problem, $f(z)$ and $g'(z)$ are regular functions of z, for every value of the time t, in the open region S, and continuous in S and on its boundaries, except possibly at points on the boundaries at which isolated forces or couples act and where the singularities are prescribed. We shall also assume that derivatives with respect to time exist up to any required order. Then, from (4.14) and (4.18) it follows that the non-dimensional stress and displacement components are single-valued and continuous throughout the interior of S for all time and that the resultant force acting on any closed contour in S is zero. Moreover, if temperatures and self-equilibrating surface tractions are prescribed on the boundaries so that the right-hand side of (4.18) is given on L_0, L_1, $\cdots$, L_n, the solution (4.14) for Θ and Φ of the resulting boundary-value problem involves only the physical parameters α and β $(1 - 2\sigma)/(1 - \sigma)$. It is thus seen that if we regard the non-dimensional stress components given by (4.14) as the actual stress components in a scale model of the body under consideration, then the stress components in the original body can be calculated from those in the scale model by using the scale relations (2.4), provided that α and $\beta(1 - 2\sigma)/(1 - \sigma)$ have the same values in the model and the original body. However, the non-dimensional displacement components D given by (4.14) involve the physical parameters α, σ and β $(1 - 2\sigma)/(1 - \sigma)$ and therefore the displacement components in the original body can, in general, be calculated from those of the scale model only if α, σ and β have the same values in the model and the original body. If the surface tractions on each of the contours L_0, L_1, $\cdots$, L_n are not self-equilibrating, then the conditions for the stress and displacement components to be single-valued are given by Eqs. (5.3). Thus, if the surface tractions on L_0, L_1, $\cdots$, L_n, given by (4.18), are specified, it is seen that Eqs. (4.14) for Θ and Φ and Eq. (5.3) involve the physical parameters α, $\beta(1 - 2\sigma)/(1 - \sigma)$ and σ. Hence, in order to calculate the stress components of a body from those of a scale model, by means of Eqs. (2.4), α, σ and β must, in general, have the same values in the body and the model.

7. Generalized plane stress. We consider a body bounded by plane faces $x_3 = \pm h$, where h is a non-dimensional constant, and by cylindrical surfaces perpendicular to to these surfaces. The planes $x_3 = \pm h$ are assumed to be free from applied stress and to be insulated so that there is no loss of heat from them. Thus,

$$\frac{\partial T}{\partial x_3} = 0, \qquad t_{31} = t_{23} = t_{33} = 0 \qquad (x_3 = \pm h). \tag{7.1}$$

We assume that the material of the body has Poisson's ratio σ^*, coefficient of thermal expansion ν^*, rigidity modulus μ^* and thermal diffusivity κ^* and α^* and β^* are defined

$$\alpha^* = \frac{\tau_0 \kappa^*}{l^2} \quad \text{and} \quad \beta^* = \frac{\Theta_0 \nu^*}{\mu^*},$$

where l is again a standard length, τ_0 a standard time and Θ_0 a standard temperature and μ^* is taken as the scale factor for the stress components.

From (2.6), replacing σ and β by σ^* and β^*, we have

$$e_{33} = -\frac{\sigma^*}{1 - \sigma^*} e_{\rho\rho} + \frac{1 - 2\sigma^*}{2(1 - \sigma^*)} (t_{33} + \beta^* T) \tag{7.2}$$

and using this to eliminate e_{33} from the remaining equations in (2.6) we find that

$$t_{\lambda\mu} = 2\left[e_{\lambda\mu} + \frac{\sigma^*}{1 - \sigma^*} e_{\rho\rho}\delta_{\lambda\mu}\right] - \frac{\beta^*(1 - 2\sigma^*)T\delta_{\lambda\mu}}{1 - \sigma^*} + \frac{\sigma^* t_{33}\delta_{\lambda\mu}}{1 - \sigma^*},$$

$$t_{\lambda 3} = 2e_{\lambda 3}. \tag{7.3}$$

We now assume that the body is subject to a stress system which is such that the displacements u_1, u_2 are even functions of x_3 and u_3 is an odd function of x_3. We then follow the usual procedure and take average values of Eqs. (2.5) and (7.3), using conditions (7.1). We add the extra assumption that the average value of t_{33} is neglected in comparison with the average values of $t_{\lambda\mu}$. Denoting the average values of $t_{\lambda\mu}$, $e_{\lambda\mu}$, u_α and T by $t^*_{\lambda\mu}$, $e^*_{\lambda\mu}$, u^*_α and T^* respectively, we obtain

$$\frac{\partial t^*_{\lambda\mu}}{\partial x_\mu} = 0, \tag{7.4}$$

$$t^*_{\lambda\mu} = 2\left(e^*_{\lambda\mu} + \frac{\sigma^*}{1 - \sigma^*} e^*_{\rho\rho}\delta_{\lambda\mu}\right) - \frac{\beta^*(1 - 2\sigma^*)}{1 - \sigma^*} T^*\delta_{\lambda\mu},$$

$$e^*_{\lambda\mu} = \frac{1}{2}\left(\frac{\partial u^*_\lambda}{\partial x_\mu} + \frac{\partial u^*_\mu}{\partial x_\lambda}\right) \tag{7.5}$$

and

$$\frac{\partial T^*}{\partial x_\lambda\, \partial x_\lambda} = \frac{1}{\alpha^*} \frac{\partial T^*}{\partial t}. \tag{7.6}$$

If we denote by X^*_α the average values of the components of the non-dimensional resultant force (per unit of x_3-axis) due to the stresses acting across an arc AB of the material lying in the x_1x_2-plane, then we see from (4.15) that

$$X^*_1 = \int_A^B (t^*_{12}\, dx_1 - t^*_{11}\, dx_2), \qquad X^*_2 = \int_A^B (t^*_{22}\, dx_1 - t^*_{12}\, dx_2). \tag{7.7}$$

We note, replacing μ by μ^* in Eqs. (2.4), that the average values of the actual stresses $\tau^*_{\lambda\mu}$, displacements v^*_α and temperature Θ^* can be obtained from the non-dimensional values by means of the relations

$$y_\alpha = lx_\alpha, \qquad v^*_\alpha = lu^*_\alpha, \qquad \tau = \tau_0 t, \qquad \tau^*_{\lambda\mu} = \mu^* t^*_{\lambda\mu},$$

and

$$\Theta^* = \Theta_0 T^*. \tag{7.8}$$

Equations (7.4) to (7.7) are identical in form with Eqs. (4.3), (4.4), (4.5) and (4.15) if we replace

$$t^*_{\lambda\mu}\,, \qquad e^*_{\lambda\mu}\,, \qquad T^*, \qquad u^*_\lambda\,, \qquad \sigma^*/(1-\sigma^*), \qquad \beta^*(1-2\sigma^*)/(1-\sigma^*)$$

and α^* in the former by $t_{\lambda\mu}$, $e_{\lambda\mu}$, T, u_λ, $\sigma/(1-2\sigma)$, β and α respectively. Hence, we can write down the solution of Eqs. (7.4) to (7.7) by analogy with the plane strain problem. The conditions for the average values of the non-dimensional displacement and stress components to be single-valued may similarly be written down by analogy with the case of plane strain.

With the notation

$$\Theta^* = t^*_{11} + t^*_{22}\,, \qquad \Phi^* = t^*_{11} - t^*_{22} + 2it^*_{12} \quad \text{and} \quad D^* = u^*_1 + iu^*_2\,, \tag{7.9}$$

we obtain [see Eqs. (4.14)]

$$\begin{aligned} \Theta^* &= 4[f^{*\prime}(z) + \bar{f}^{*\prime}(\bar{z})] - \beta^*(1-2\sigma^*)T^*, \\ \Phi^* &= -4[z\bar{f}^{*\prime\prime}(\bar{z}) + \bar{g}^{*\prime\prime}(\bar{z})] + 4\beta^*(1-2\sigma^*)\frac{\partial^2 R^*}{\partial\bar{z}\,\partial\bar{z}}\,, \\ D^* &= \frac{3-\sigma^*}{1+\sigma^*} f^*(z) - z\bar{f}^{*\prime}(\bar{z}) - \bar{g}^{*\prime}(\bar{z}) + \beta^*(1-2\sigma^*)\frac{\partial R^*}{\partial\bar{z}}\,, \end{aligned} \tag{7.10}$$

where $f^*(z)$ and $g^*(z)$ are functions of z which depend on the boundary conditions and $R^*(z, \bar{z}, t)$ is a real particular integral of the equation

$$4\frac{\partial^2 R^*}{\partial z\,\partial\bar{z}} = T^*. \tag{7.11}$$

From (7.7) and (7.10) it is seen that

$$X^*_1 + iX^*_2 = 2i\left[f^*(z) + z\bar{f}^{*\prime}(\bar{z}) + \bar{g}^{*\prime}(\bar{z}) - \beta^*(1-2\sigma^*)\frac{\partial R^*}{\partial\bar{z}}\right]_A^B. \tag{7.12}$$

It can readily be seen, in a manner similar to that adopted in Sec. 5, that the conditions for the average values of the non-dimensional average stress and average displacement components $t^*_{\lambda\mu}$ and u^*_λ to be single-valued are

$$[f^{*\prime}(z)]_A^A = 0, \qquad [g^{*\prime\prime}(z)]_A^A = 0$$

and

$$\frac{3-\sigma^*}{1+\sigma^*}[f^*(z)]_A^A = [\bar{g}^{*\prime}(\bar{z})]_A^A\,. \tag{7.13}$$

In the case when the stress system is such that the resultant average force acting on any closed curve in the body is zero, the conditions (7.13) become

$$[f^*(z)]_A^A = 0, \qquad [g^{*\prime}(z)]_A^A = 0. \tag{7.14}$$

8. Similarity laws for generalized plane stress. We now assume, as in the case of plane strain previously discussed, that the cross-section of the body considered, normal to the x_3-axis, is finite and multiply-connected and bounded by one or more smooth non-intersecting contours L_0, L_1, $\cdots$, L_n, of which L_0 contains all the others, and that the values of the non-dimensional average surface tractions (and hence of X^*_α) and

non-dimensional average temperature T^* are given over each of the contours L_0 , L_1 , $\cdots$, L_n . Since, from Eqs. (7.6) and (7.11), the values of T^* and R^* throughout the body depend only on the physical parameter α^*, the expressions (7.10) and (7.12) for Θ^*, Φ^* and X_α^* involve only the physical parameters α^* and $\beta^*(1 - 2\sigma^*)$. Also, if the surface tractions applied to each of the contours L_0 , L_1 , $\cdots$, L_n are self-equilibrating, the conditions (7.14) for the average stress and average displacement components to be single-valued do not involve any physical constants. It follows that, in the case when the specified surface tractions applied to each of the contours L_0 , L_1 , $\cdots$, L_n are self-equilibrating, if the non-dimensional average stress distribution is taken as the actual average stress distribution in a scale model of the body under consideration, then the average stress distribution in the original body can, in general, be calculated from that in the model by employing the relations (7.8), if α^* and $\beta^*(1 - 2\sigma^*)$ have the same values for the model and the original body. In the case when the specified surface tractions applied to each of the contours L_0 , L_1 , $\cdots$, L_n are not self-equilibrating, since the conditions (7.13) for the stress and displacement components to be single-valued involve σ^*, the average stress distribution in the original body can, in general, be obtained from that in the model by means of the relations (7.8) only if α^*, σ^* and β^* have the same values in the model and the original body.

Now, suppose that we have two bodies A and B with geometrically similar cross-sections bounded by the closed contours L_0 , L_1 , $\cdots$, L_n , as described above. Let the dimensions of the cross-section of A be l times those of B. The body A is maintained in a state of plane thermo-elastic strain and B is maintained in a state of generalized plane thermo-elastic stress by surface tractions and surface temperatures such that the actual average surface tractions and average surface temperatures in B are equal to the non-dimensional surface tractions and surface temperatures in A. We assume that the physical parameters for the body A are the unstarred parameters defined in Sec. 2, while those for the body B are the starred parameters defined in the present section. Taking $l = 1$, $\tau_0 = 1$, $\mu^* = 1$ and $\Theta_0 = 1$ for the body B, we see that (7.10) and (7.12) give the actual average stress components and surface tractions in B, while T^* represents the actual average temperature in B. Then, comparing Eqs. (7.6), (7.10), (7.11), (7.12) and (7.13) with Eqs. (4.5), (4.14), (4.13), (4.18) and (5.3), and bearing in mind that we have assumed $T = T^*$ and $X_\alpha = X_\alpha^*$ on the surfaces of A and B, we see that provided that

$$\alpha^* = \alpha, \qquad \beta^*(1 - 2\sigma^*) = \frac{\beta(1 - 2\sigma)}{1 - \sigma}, \qquad \frac{3 - \sigma^*}{1 + \sigma^*} = 3 - 4\sigma, \tag{8.1}$$

i.e. provided that

$$\alpha^* = \alpha, \qquad \sigma^* = \frac{\sigma}{1 - \sigma} \quad \text{and} \quad \beta^* = \frac{\beta(1 - 2\sigma)}{1 - 3\sigma}, \tag{8.2}$$

we have

$$\Theta^* = \Theta \quad \text{and} \quad \Phi^* = \Phi, \tag{8.3}$$

so that the stress distribution in A can be calculated from the average stress distribution in B by the scale factors defined in Eqs. (2.4).

If the surface tractions applied to each of the contours L_0 , L_1 , $\cdots$, L_n are self-equilibrating, then comparing Eqs. (7.6), (7.10), (7.11), (7.12) and (7.14) with Eqs.

(4.5), (4.14), (4.13), (4.18) and (5.4), we see that Eqs. (8.3) are satisfied provided that

$$\alpha^* = \alpha \quad \text{and} \quad \beta^*(1 - 2\sigma^*) = \frac{\beta(1 - 2\sigma)}{1 - \sigma}. \tag{8.4}$$

9. Appendix. The function R. In deriving in Sec. 5 the conditions for the stress and displacement components to be single-valued the assumption was made that the function $R(z, \bar{z}, t)$ satisfying Eq. (4.13) can be chosen to be real and single-valued, together with its derivatives up to the fourth order. This assumption will be justified in the present section.

We deal first with the case of steady heat flow and statical equilibrium for which $T = T_0(z, \bar{z})$, $R = R_0(z, \bar{z})$ and

$$\frac{\partial^2 T_0}{\partial z\, \partial \bar{z}} = 0, \qquad T_0 = 4\, \frac{\partial^2 R_0}{\partial z\, \partial \bar{z}}. \tag{9.1}$$

Since T_0 is real, the first of Eqs. (9.1) may be integrated in the form

$$T_0 = h'(z) + \bar{h}'(\bar{z}), \tag{9.2}$$

where, since T_0 and its derivatives are single-valued inside the region S occupied by the cross-section of the body considered,

$$[h''(z)]_A^A = 0, \qquad [h'(z) + \bar{h}'(\bar{z})]_A^A = 0. \tag{9.3}$$

We now suppose that the region S is the multiply-connected region bounded by contours $L_0, L_1, \cdots, L_n$, as described in Sec. 6. We assume that the temperature distribution in the body is such that $h''(z)$ [which is seen from (9.3) to be single-valued] is a regular function of z in the open region S. From Laurent's theorem it can be seen that $h''(z)$ must be expressible in the form

$$h''(z) = \sum_{k=1}^{n} \frac{a_k}{2\pi(z - z_k)} - \sum \frac{b_k + id_k}{2\pi(z - z_k)^2} + k''(z), \tag{9.4}$$

where $k(z)$ is a regular function of z in S, b_k and d_k are real constants, z_k is a point inside the contour L_k (and therefore outside S) and a_k is a constant. The result (9.4) follows from the fact that if terms of higher degree than the second in $1/(z - z_k)$ are included in the Laurent expansion (9.4) for $h''(z)$ they can be absorbed into the function $k''(z)$ without vitiating the regularity of $k(z)$ in the region S.

By integration of (9.4), we obtain

$$h'(z) = \sum_{k=1}^{n} \frac{a_k}{2\pi} \log(z - z_k) + \sum_{k=1}^{n} \frac{b_k + id_k}{2\pi(z - z_k)} + k'(z), \tag{9.5}$$

$$\begin{aligned} h(z) = \sum_{k=1}^{n} \frac{a_k}{2\pi}(z - z_k)\log(z - z_k) + \sum_{k=1}^{n} \frac{b_k + id_k}{2\pi} \log(z - z_k) \\ + k(z) - \sum_{k=1}^{n} \frac{a_k}{2\pi}(z - z_k). \end{aligned} \tag{9.6}$$

From the second of Eqs. (9.3), it is seen that a_k is a real constant. It is apparent from (9.2) and (9.5) that if $h(z)$ is expressible in the form (9.6) with a_k real, T_0 is a real, single-valued function in S.

Introducing the expression (9.2) into the second of Eqs. (9.1), we obtain

$$4 \frac{\partial^2 R_0}{\partial z \, \partial \bar{z}} = h'(z) + \bar{h}'(\bar{z}). \tag{9.7}$$

Introducing the expression (9.5) for $h'(z)$ into (9.7), it is apparent that (9.7) has a particular integral for R_0 of the form

$$\begin{aligned} 4R_0 = {} & \sum_{k=1}^{n} \frac{a_k}{2\pi} (z - z_k)(\bar{z} - \bar{z}_k)[\log (z - z_k) + \log (\bar{z} - \bar{z}_k)] \\ & + \sum_{k=1}^{n} \left[\frac{b_k + id_k}{2\pi} \bar{z} + \frac{b_k - id_k}{2\pi} z \right] [\log (z - z_k) + \log (\bar{z} - \bar{z}_k)] + l(z, \bar{z}), \end{aligned} \tag{9.8}$$

where $l(z, \bar{z})$ is a real single-valued function in S, given by

$$l(z, \bar{z}) = \bar{z}k(z) + z\bar{k}(\bar{z}) - \sum_{k=1}^{n} \frac{a_k}{\pi} z\bar{z}. \tag{9.9}$$

Since the expression on the right-hand side of (9.8) is real and it and its derivatives are single-valued, R_0 may be chosen in such a way that it is real and it and its derivatives are single-valued.

When the heat flow is not steady, T and R satisfy the equations

$$4 \frac{\partial^2 T}{\partial z \, \partial \bar{z}} - \frac{1}{\alpha} \frac{\partial T}{\partial t} = 0, \qquad 4 \frac{\partial^2 R}{\partial z \, \partial \bar{z}} = T. \tag{9.10}$$

Eliminating T, we obtain

$$\frac{\partial^2}{\partial z \, \partial \bar{z}} 4 \left(\frac{\partial^2 R}{\partial z \, \partial \bar{z}} - \frac{1}{\alpha} \frac{\partial R}{\partial t} \right) = 0. \tag{9.11}$$

Equation (9.11) is satisfied by

$$4 \frac{\partial^2 R}{\partial z \, \partial \bar{z}} - \frac{1}{\alpha} \frac{\partial R}{\partial t} = Q, \tag{9.12}$$

where Q is any real function which satisfies the equation

$$\frac{\partial^2 Q}{\partial z \, \partial \bar{z}} = 0. \tag{9.13}$$

We make a particular choice of the function Q that it be expressible in the form

$$Q = -\frac{1}{\alpha} \frac{\partial P}{\partial t} + T_0 , \tag{9.14}$$

where T_0 is a real single-valued function, independent of t, which satisfies the equation

$$\frac{\partial^2 T_0}{\partial z \, \partial \bar{z}} = 0 \tag{9.15}$$

and P is a real function which satisfies the equation

$$\frac{\partial^2 P}{\partial z \, \partial \bar{z}} = 0. \tag{9.16}$$

Since T_0 satisfies (9.15) and is real and single-valued, we see, as we did in discussing the function T_0 given by the first of Eqs. (9.1), that there exists* a function $R_0(z, \bar{z})$ which is real and single-valued together with its derivatives up to the fourth order and satisfies the equation

$$T_0 = 4 \frac{\partial^2 R_0}{\partial z\, \partial \bar{z}}. \tag{9.17}$$

Defining $R_1(z, \bar{z}, t)$ by

$$R = P + R_0 + R_1 , \tag{9.18}$$

we see, from (9.12), (9.14), (9.16), (9.17) and (9.18), that R_1 must satisfy the equation

$$4 \frac{\partial^2 R_1}{\partial z\, \partial \bar{z}} - \frac{1}{\alpha} \frac{\partial R_1}{\partial t} = 0. \tag{9.19}$$

Substituting from (9.18) in the second of Eqs. (9.10) and employing (9.16) and (9.17), we have

$$T = T_0 + T_1 , \tag{9.20}$$

where

$$T_1 = 4 \frac{\partial^2 R_1}{\partial z\, \partial \bar{z}}. \tag{9.21}$$

From (9.19) and (9.21)

$$T_1 = \frac{1}{\alpha} \frac{\partial R_1}{\partial t}. \tag{9.22}$$

Since T and T_0 are real and single-valued, we see from (9.20) that T_1 must be a real single-valued function of z and $\bar{z}$ for every t. It follows from (9.22) that we may choose R_1 to be a real single-valued function of z and $\bar{z}$ for all t. It must, of course, also be single-valued in t and possess derivatives with respect to z, $\bar{z}$ and t of any required order. We have already seen that R_0 can be chosen to be real and single-valued and P has been chosen to be real and single-valued. It follows from (9.18) that R is real and single-valued.

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*In deriving this result, we note that in this case also T_0 must have the form (9.2) and we assume as in the earlier discussion that T_0 is such that $h''(z)$ is a regular function of z in the open region S.

BOOK REVIEWS

(*Continued from p. 340*)

Handbuch der Laplace-Transformation. By Gustav Doetsch. Volume II. Verlag Birkhäuser, Basel and Stuttgart, 1955. 436 pp. $13.15.

This book is the second volume of the author's trilogy on the Laplace transformation. Volume I, which appeared in 1950, was concerned with the theory of the Laplace transformation. Volumes II and III are dedicated to the applications of the Laplace transformation in various domains of pure and applied mathematics. The present volume is divided into three parts, entitled: Asymptotic expansions, Convergent expansions, and Ordinary differential equations. An introductory chapter collects together the main results and "rules" concerning the one sided Laplace transformation $f(s) = \int_0^\infty e^{-st}F(t)\,dt$ and the two sided Laplace transformation $f(s) = \int_{-\infty}^{\infty} e^{-st}\,F(t)\,dt$. The first part centers around the following general definition of an asymptotic expansion ("im Sinne von Poincaré"): a function $g(z)$ is said to have an asymptotic expansion $\sum_{j=0}^{\infty} c_j h_j(z)$ in a neighborhood U of z_0 provided that for each $n = 0, 1, \ldots$ one has $g(z) - \sum_{j=0}^{n} c_j h_j(z) = o(h_n(z))$ as $z \to z_0$. (It is supposed that $h_n(z) \neq 0$ in U.) Given any functional transformation T, (so that $T(F) = f$, say), a theorem which concludes something about the asymptotic behavior of the image function f from a knowledge of the asymptotic behavior of the original function F is designated as an "Abelsche Asymptotik". If the rôles of the image function and the original function are interchanged in the last statement, then such a theorem is known as a "Taubersche Asymptotik". The main portion of the first part is dedicated to the Abelsche Asymptotik of the one and two sided Laplace transformations, the Mellin transformation, and the complex inversion integral of the Laplace transformation (this last is of particular importance in technical applications, when the properties of a function defined by an inversion integral have to be determined). There is also a chapter on the Taubersche Asymptotik. The second part devoted to convergent expansions, consists of two chapters, the first on faculty series

$$\sum_{n=0}^{\infty} \frac{a_n n!}{s(s+1)\cdots(s+n)},$$

and the second on various series expansions, e.g. theta function, Laguerre polynomials, and others. The main concern here is with convergent series which are obtained by term by term application of the Laplace transformation. The first chapter of the third part deals with the ordinary differential equation with constant coefficients (which may be written in symbolic form thus: $p(d/dt)Y = F$, where $p(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_1x + c_0$ and $F = F(t)$) on the interval $0 < t < \infty$, the initial conditions being

$$Y_0 = \lim_{t\to+0} Y(t), \cdots, Y_0^{(n-1)} = \lim_{t\to+0} Y^{(n-1)}(t),$$

while the next chapter considers the same differential equation on the infinite interval $-\infty < t < +\infty$ with various boundary conditions at $-\infty$ and $+\infty$. There are detailed applications to the theory of wave filters and electrical networks, among others. The system of ordinary differential equations with constant coefficients

$$\begin{gathered} p_{11}(D)Y_1 + p_{12}(D)Y_2 + \cdots + p_{1N}(D)Y_N = F_1(t) \\ \cdots \qquad \cdots \qquad \cdots \\ p_{N1}(D)Y_1 + p_{N2}(D)Y_2 + \cdots + p_{NN}(D)Y_N = F_N(t), \end{gathered}$$

where

$$p_{jk}(s) = c_n^{jk}s^n + \cdots + c_1^{jk}s + c_0^{jk} \qquad (j, k = 1, \cdots, N)$$

is also treated (specially in the "Normalfall"). The remaining chapters develop the theory of ordinary differential equations with variable coefficients. As is his custom, the author has included a large number of valuable literary and historical remarks at the end of the volume. This book is remarkable for the wealth of detail and the clarity of the exposition.

J. B. Diaz

(*Continued on p. 435*)

THE RECTIFICATION OF NON-GAUSSIAN NOISE*

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Section I. Introduction. This paper considers the effects of a general class of non-gaussian random processes in an *a-m* receiving system. Analysis of the effects of non-gaussian noise is needed both for use in problems where the noise is known not to be gaussian, e.g. some kinds of radar clutter and atmospheric static, and to indicate in uncertain cases how critical the assumption of normal statistics may be.

The probability densities of a generalized Poisson process and a new approximating series for the densities are presented in Sec. II, with discussion of and results for the rectification problem following in Sec. III.

The asymptotic approximation to the probability distributions, using derivatives with respect to the second moments, is new in its general form, although foreshadowed by some special results obtained earlier by Edgeworth [1] and Pearson [2]. The problem of rectification of non-gaussian noise does not appear to have been attacked with reasonable generality before. A number of papers have dealt with the deviation of rectifier outputs from normal statistics when the input is normally distributed [3, 4, 5] and some results are available for non-gaussian noise in quadratic detectors [6] and for very narrow post detector filters [7]. We have obtained expressions for the output covariance function of a half wave νth-law rectifier with a sine wave carrier and non-gaussian noise input. Explicit results are presented for linear and quadratic detectors. A qualitative discussion of the behavior of rectified non-gaussian noise for general values of ν is also included.

In general, the work has shown that the difference between the effects of gaussian and non-gaussian noise of the same input power is fairly small unless the signal is weak or the noise is strongly non-gaussian. Usually non-gaussian noise produces a greater output intensity and a broader output spectrum than gaussian noise with the same input power and input spectrum.

Section II. Probability densities. 2.1 Introduction. The probability concepts and notation used here are briefly presented in this section. Probability densities are used rather than cumulative distribution functions; in the usual way, for example, one writes $W_2(y_1, y_2; t_2 - t_1)\, dy_1\, dy_2 =$ the joint probability for a stationary random process that y will lie in the interval $(y_1, y_1 + dy_1)$ at time t_1, and in the interval $(y_2, y_2 + dy_2)$ at time t_2.

The distribution can also be characterized by its moments (whenever these exist), e.g.(*)

$$\mu_{mn}(t) = \langle y_1^m y_2^n \rangle = \iint_{-\infty}^{\infty} y_1^m y_2^n W_2(y_1, y_2; t)\, dy_1\, dy_2 . \tag{1}$$

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(*)The angle brackets here and subsequently denote the statistical average.

The Fourier transform of the probability density is the characteristic function,

$$F_{2y}(\xi_1, \xi_2; t) = \langle \exp (i\xi_1 y_1 + i\xi_2 y_2) \rangle, \tag{2}$$

whose power series expansion generates the moments,

$$F_2(\xi_1, \xi_2; t) = \sum_{m,n=0}^{\infty} \frac{(i\xi_1)^m (i\xi_2)^n}{m!\, n!} \mu_{mn}(t), \tag{3}$$

(again, whenever $\mu_{mn}(t)$ exist). An equally useful set of parameters are the semi-invariants, $\lambda_{mn}(t)$, defined by

$$\log F_2(\xi_1, \xi_2; t) = \sum_{m,n=1}^{\infty} \frac{(i\xi_1)^m (i\xi_2)^n}{m!\, n!} \lambda_{mn}(t). \tag{4}$$

2.2 The Poisson ensemble. The non-gaussian random processes which we shall use are Poisson ensembles, i.e. they are composed of sums of independent variables with common distributions and uniformly distributed times of occurrence, t_i', viz.,

$$V(t; K, \{t_i'\}) = \sum_{i=1}^{K} v_i(t - t_i'). \tag{5}$$

Physically, the v_i represent pulses produced randomly by a noise source. The above assumptions imply that the number of pulses, K, occurring in a long time interval, T, is a random variable with a Poisson distribution [8] so that K, as well as the set of t_i', is an ensemble parameter.

The characteristic function of the process, for a finite interval of time T, can be readily derived by taking advantage of the independence of the pulses and performing a weighted sum over a random walk for each fixed K [9, 10]. Thus we write

$$\begin{aligned} F_{2V}(\xi_1, \xi_2; t)_T &= \sum_{K=0}^{\infty} \frac{(nT)^K}{K!} e^{-nT} [F_{2v}(\xi_1, \xi_2; t)_T]^K, \\ &= \exp \{nT[F_{2v}(\xi_1, \xi_2; t)_T - 1]\}, \end{aligned} \tag{6}$$

where n is the average number of pulses occurring per unit time. F_{2v} is the characteristic function for the variable v, representing an individual pulse, and F_{2V} is the characteristic function for the sum, V.

The finiteness of the time interval can be eliminated by first expressing the exponent as an average,

$$nT[F_{2v}(\xi_1, \xi_2; t)_T - 1] = nT\langle \exp (i\xi_1 v_1 + i\xi_2 v_2) - 1 \rangle, \tag{7}$$

in which the average must be carried out over all the random variables of a typical pulse, e.g. amplitude, shape, phase, duration, or time of occurrence. Let us maintain the average sign as a reminder of the presence of other random variables and explicitly consider the averages for the times of occurrence. If the time variables are transformed from t_1', t_2' to $t - t_1'$, $t_2' - t_1'$, then, taking advantage of the uniform distribution of the occurrence time, we have

$$W(t_1', t_2')\, dt_1'\, dt_2' = W(t_2' - t_1') \frac{1}{T} \tau\, d\left(\frac{t - t_1'}{\tau}\right) d(t_2' - t_1'), \tag{8}$$

where τ is the mean duration of a pulse, so that letting T increase indefinitely, we get

$$F_{2V}(\xi_1, \xi_2; t) = \exp\left\{\gamma \int_{-\infty}^{\infty} \langle\{\exp[i\xi_1 v_1(t_0) + i\xi_2 v_2(t_0 + t)]\} - 1\rangle\, d\left(\frac{t_0}{\tau}\right)\right\}, \tag{9}$$

where $\gamma = n\tau$ is the average noise "density," i.e. the average number of pulses per second multiplied by the average duration of a pulse.

The semi-invariants of V are readily obtained by expanding the inner exponential of Eq. (9), yielding

$$\lambda_{mn}(t) = \gamma\langle v(t_0)^m v(t_0 + t)^n\rangle. \tag{10}$$

One is at once tempted to say, from Eq. (10), that the semi-invariants of V are proportional to the moments of v. This is not entirely proper, because the presence of the -1 under the average in Eq. (9) acts as a convergence factor, so that the exponent is not a true characteristic function, and hence the right hand side of Eq. (10) is not a true moment. For a finite time interval, no convergence problems arise, and Eq. (6) shows that then indeed the semi-invariants of V are proportional to the moments of v.

The parameter of the Poisson ensemble which has most influence in determining the general features of the noise is the density γ. As the noise density increases, the noise distributions tend toward the gaussian. When the density is small, the noise pulses overlap only slightly so that the noise has the attributes of a deterministic, interfering signal, whose time-structure is essentially that of a single typical impulse. These two limiting regions are important, since the exact distributions are usually too complex to be used analytically, and must be approximated differently in each of the two cases.

2.3 Narrow band noise. Here, in the course of rectification, the detector or rectifier is preceded by a frequency selective network which passes only a band of frequencies narrow compared to the center frequency of the band. This is simultaneously desirable in order to discriminate against unwanted signals in other frequency bands, and more or less inescapable because of the inherent characteristics of the elements of which the receiver is built. Thus, although the noise at the input of the receiver may be, and usually is, of relatively constant strength over a wide band of frequencies, the noise presented to the detector is narrow-band because of its passage through the frequency selective parts of the receiver.

Important simplifications of the probability densities can be obtained by taking explicit cognizance of the narrow-band nature of the processes with which we are concerned. A narrow-band random variable can be expressed as

$$V(t) = R(t) \cos[\omega_0 t - \theta(t)], \tag{11}$$

where ω_0 is the central frequency of the noise spectrum and R and θ are random variables whose variation with time is slow compared to that of $\cos \omega_0 t$. Thus, if one calculates the moments of V using time averages (over a single member function of the V ensemble, which is assumed to be ergodic), R and θ can be assumed to be constant over a single period of $\cos \omega_0 t$. Consequently, all the moments of V which are of odd degree vanish. The constancy of envelope and phase over one period of the center frequency is, of course, an approximation and cannot be expected to be valid for moments of arbitrarily high degree, since a change in R of δR will give a change in R^n of $nR^{n-1}\,\delta R$, and thus, for sufficiently large n, will be comparable to the change in $\cos \omega_0 t$. In the approximating distributions which will be used, however, only low-degree moments will appear, so that the invariance of the slowly varying parts over a single period of $\cos \omega_0 t$ is a valid assumption.

The presence of high frequency terms, such as $\cos \omega_0 t$, in the expression for the second-order moments is not eliminated, by any means. In order to indicate clearly the presence of high frequency terms, let us define an envelope factor for the moments, by

$$M_{mn}(t) \equiv \frac{\Gamma([m+n+1]/2)}{\Gamma(1/2)\Gamma([m+n]/2+1)} \langle R_1^m R_2^m \rangle. \tag{12}$$

Let us further assume that the phase change of the noise voltage, i.e. $\theta_2 - \theta_1$, is not a random variable*. This is not a valid restriction on some types of random waves; but, in this paper, we shall need the results below in their application to individual pulses [the v_i of Eq. (5)], where the phase would be expected to be constant.

The second-order moments can now be written

$$\begin{aligned} \mu_{11}(t) &= \langle R_1 \cos(\omega_0 t_0 - \theta_1) R_2 \cos(\omega_0 t_0 + \omega_0 t - [\theta_1 + \alpha])\rangle \\ &= \langle R_1 R_2 \cos^2(\omega_0 t_0 - \theta_1)\rangle \cos(\omega_0 t - \alpha) \\ &\quad - \langle R_1 R_2 \cos(\omega_0 t_0 - \theta_1)\sin(\omega_0 t_0 - \theta_1)\rangle \sin(\omega_0 t - \alpha) \\ &= \tfrac{1}{2}\langle R_1 R_2\rangle \cos(\omega_0 t - \alpha) \\ &= M_{11}(t)\cos(\omega_0 t - \alpha), \end{aligned} \tag{13a}$$

and similarly

$$\mu_{31}(t) = M_{31}(t)\cos(\omega_0 t - \alpha), \tag{13b}$$

$$\mu_{22}(t) = M_{22}(t)[\tfrac{1}{3} + \tfrac{2}{3}\cos^2(\omega_0 t - \alpha)], \tag{13c}$$

which, with $\mu_{m,n}(0) = \mu_{m+n,0}$, defines all the non-zero moments up to the sixth degree.

We note that by comparing the two expansions of the characteristic function, Eqs. (3) and (4), a similar separation of high and low frequency factors can be accomplished for the semi-invariants, e.g.

$$\lambda_{11}(t) = \Lambda_{11}(t)\cos(\omega_0 t - \alpha), \tag{14}$$

and that a special notation is usually employed for the autovariance function, i.e.

$$\mu_{11}(t) = R(t) = \psi r(t) = \psi r_0(t)\cos(\omega_0 t - \alpha), \tag{15}$$

where $r(0) = r_0(0) = 1$.

2.4 Nearly gaussian distributions. A nearly normal distribution, e.g. Eq. (9) when γ is large, can be expanded asymptotically in terms of the limiting gaussian form. For the case where the odd degree semi-invariants are zero, the leading terms of the expansion of the characteristic function are

$$\begin{aligned} F_2(\xi_1, \xi_2; t) = \left\{1 + \frac{\lambda_{40}}{4!}\xi_1^4 + \frac{\lambda_{31}(t)}{3!}\xi_1^3\xi_2 + \frac{\lambda_{22}(t)}{2!\,2!}\xi_1^2\xi_2^2 + \frac{\lambda_{13}(t)}{3!}\xi_1\xi_2^3 + \frac{\lambda_{04}}{4!}\xi_2^4 + \cdots\right\} \\ \cdot \exp\left\{-\frac{\psi}{2}[\xi_1^2 + 2\xi_1\xi_2 r(t) + \xi_2^2]\right\}. \end{aligned} \tag{16}$$

In taking the Fourier transform of this equation, we note that the multiplications by ξ_i will become differentiations with respect to V_i, and the two-dimensional form of the well-known Edgeworth series is obtained. The terms containing the fourth degree semi-

*Formulae for the case where the phase change is random may be found in Mullen and Middleton [11].

invariants can be shown to be of order $\Lambda_{40}/\psi^2 \sim \gamma^{-1}$ and the neglected terms to be of order γ^{-2}.

An alternative form, whose derivation is given in Appendix A, can be obtained in which derivatives are taken with respect to the second moments,

$$F_2(\xi_1, \xi_2; t) = [1 + L + 0(\gamma^{-2})] \cdot \exp \{-\tfrac{1}{2}[\psi_1\xi_1^2 + 2\xi_1\xi_2\psi r_0(t) \cos(\omega_0 t - \alpha) + \psi_2\xi_2^2]\} \big|_{\psi_1=\psi_2=\psi},$$

where

$$L = \frac{\Lambda_{40}}{3!}\frac{\partial^2}{\partial\psi_1^2} + \frac{2}{3!}\Lambda_{31}(t)\frac{\partial^2}{\partial\psi_1\,\partial\psi r_0} + \frac{\Lambda_{22}(t)}{3!}\left[\frac{\partial^2}{\partial\psi r_0^2} + 2\frac{\partial^2}{\partial\psi_1\psi_2}\right] + \frac{2\Lambda_{13}(t)}{3!}\frac{\partial}{\partial\psi_2\psi r_0} + \frac{\Lambda_{04}}{3!}\frac{\partial^2}{\partial\psi_2^2}, \tag{17}$$

and subscripts have been placed on the second moments to enable each term in the exponent to be differentiated separately.

Only semi-invariants of low degree appear, so that the narrow-band approximation is still valid. Equation (17) is applicable to any narrow-band, nearly normal distribution, whether derived from the Poisson ensemble or not.

For moments after a non-linear operation, the comparative simplicity in the analysis when the noise possesses a gaussian distribution is a strong point in favor of the series in parametric derivatives. Applying the differential operator to the result for (nonstationary) gaussian noise is likely to be involved, but is certainly straightforward. Furthermore, where the narrow-band structure of the output is important, this characteristic function possesses the additional advantage of presenting the high-frequency part only once (in the exponent) while the Edgeworth form has high-frequency terms of different orders scattered about among the various semi-invariants, as well.

2.5 Low-density distributions. When the density is small, the exact form of the distribution is too difficult to use, and at the same time, the Edgeworth type of approximation is no longer applicable. However, a useful ascending power series in γ can be obtained, starting from Eq. (9). Let

$$f(\xi_1, \xi_2; t) = 1 + \int_{-\infty}^{\infty} \langle \exp(i\xi_1 v_1 + i\xi_2 v_2) - 1\rangle\, d\left(\frac{t}{\tau}\right), \tag{18}$$

so that

$$F_{2V}(\xi_1, \xi_2; t) = \exp\{\gamma[f(\xi_1, \xi_2; t) - 1]\} = e^{-\gamma}\sum_{n=0}^{\infty}\frac{\gamma^n}{n!}f(\xi_1, \xi_2; t)^n, \tag{19}$$

a form which much resembles Eq. (6).

The utility of Eq. (19) lies in the fact that the first two terms can be used in their exact form and the remaining terms are similar to nearly gaussian distributions, where n corresponds to the noise density. After considerable algebra, one obtains

$$\begin{aligned} F_{2V}(\xi_1, \xi_2; t) = e^{-\gamma} + \gamma e^{-\gamma}\cdot f(\xi_1, \xi_2; t) + \sum_{n=2}^{\infty}\frac{\gamma^n}{n!}e^{-\gamma}\Bigg\{\left[1 - \frac{\psi^2}{2n}\cdot\frac{\partial^2}{\partial\psi^2}\right] \\ \cdot\exp\left\{-\frac{n\psi}{2\gamma}[\xi_1^2 + 2\xi_1\xi_2 r_0(t)\cos(\omega_0 t - \alpha) + \xi_2^2]\right\} + \frac{\gamma L}{n} \\ \cdot\exp\left\{-\frac{n}{2\gamma}[\psi_1\xi_1^2 + 2\psi\xi_1\xi_2 r_0(t)\cos(\omega_0 t - \alpha) + \psi_2\xi_2^2]\right\}\Bigg|_{\psi=\psi_1=\psi_2}\Bigg\}, \end{aligned} \tag{20}$$

where L is the differential operator that appears in Eq. (17). Since our aim is to obtain the covariance function after rectification, which in our representation of the rectifier (*vide* Eq. (22) *infra*), is itself a moment of the input distribution, although in general not one of integral degree, an Edgeworth series in which the low-degree moments are reproduced exactly is appropriate. As a fit to the distribution itself, however, the approximation cannot be expected to be nearly so good.

2.6 Example. The first-order probability distribution of rectangular *c-w* pulses will serve to illustrate the behavior of the distributions of the process as the density is varied.

The distributions corresponding to the terms of fixed n in Eq. (19) have been previously calculated [12]*. From these, one can easily obtain results, which are shown in Fig. 1, for the Poisson ensemble together with a nearly gaussian approximation for $\gamma = 1$ and a gaussian distribution. The figure is a plot of the probability that the value of the abscissa, measured in units of the standard deviation, will be exceeded in absolute value.

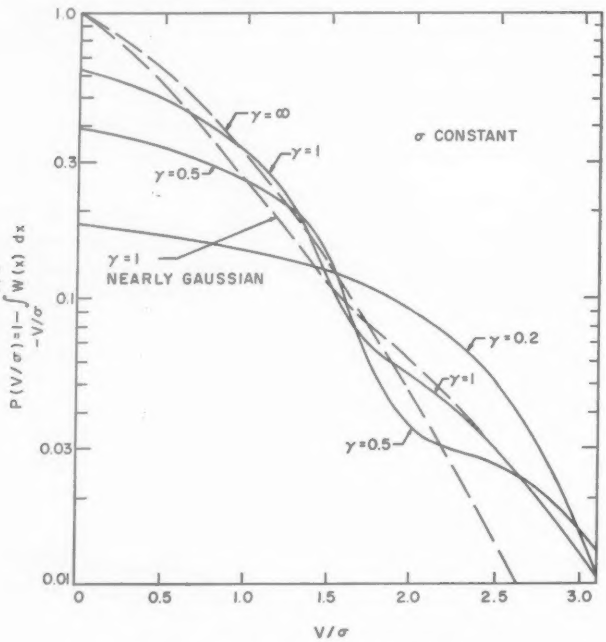

FIG. 1. Non-gaussian probability distributions.

When the noise density is one, the nearly gaussian approximation is a fairly close fit to the tail of the distribution, but is significantly different for small values of V. With the Poisson ensemble there is a finite probability of having no noise at all, so that

*Just as in Eq. (10) the right-hand side is not properly a moment, so in Eq. (19), f is not properly a characteristic function. However, the appearance of the anomalous behavior is entirely in the time structure of the process so that, for the first-order distribution, each term of the power series in γ can be considered as a characteristic function.

the non-gaussian distribution should contain a δ-function of strength $e^{-\gamma}$ at the origin. The nearly gaussian expansion, because of its asymptotic character, is unable to represent discrete probabilities.*

When plotted against V, the low-density curves lie above one another; but dividing this abscissa by the standard deviation, which varies as $\gamma^{1/2}$, causes the curves to intersect. As the density decreases, the probability that the noise will be zero becomes larger. This in turn decreases the standard deviation faster than it does the tail of the distribution, so that the probability of relatively high values increases, for smaller noise densities, relative to the larger values of noise density.

Section III. Properties of the rectifier output. 3.1 Introduction. Our next task is to use the distributions found in Sec. II to determine the autovariance function of the output of an *a-m* receiver when an additive mixture of *c-w* signal and non-gaussian noise is impressed on the input, viz.;

$$V_{IN}(t) = V_N(t) + A_0 \cos \omega_0 t. \tag{21}$$

The receiver in an amplitude modulation system contains a band-pass filter which eliminates all spectral components except those in a narrow-band around the central frequency to which it is tuned, and a demodulator, consisting of a half-wave νth-law detector followed by a low-pass filter. The relation between output and input of a non-linear device is the dynamic transfer characteristic. For the half-wave (zero memory) νth-law device, it is

$$I = g(V) = \begin{cases} \beta V^{\nu}, & V > 0 \\ 0, & V < 0 \end{cases}, \tag{22}$$

where I is the instantaneous output, V the instantaneous input, and β is an appropriate proportionality constant. The non-linear device produces an output whose spectral components lie in zones around multiples of the central frequency of the input [14]; the (ideal) low-pass filter eliminates all these zones except the low-frequency one corresponding to the zeroth harmonic.

We wish now to find the autovariance function (or since the input process is assumed to be ergodic, the correlation function) of the output. This is by no means the most complete statistical description that one could wish for, although the correlation function yields considerable insight into the nature of the output. However, to obtain the probability distributions of the low-frequency zone is a very difficult problem, which has been solved only for the quadratic detector with gaussian input noise (since no moments higher than the fourth order are then required) [15, 16].

Some general statements about the output covariance function can be made immediately. The covariance function equals the mean square of the output when its argument is zero. As the argument, t, increases beyond bounds, the two functions to be averaged become uncorrelated, so that the covariance function becomes the square of the output mean. The behavior peculiar to $R(t)$ is revealed by its variations with time, which can best be separated by defining a normalized output covariance function

$$r_{\text{out}}(t) \equiv \frac{R(t) - R(\infty)}{R(0) - R(\infty)}. \tag{23}$$

*A similar situation arises in the discussion of the Brownian motion, where a δ-function appears in the exact form of the velocity distribution which the usual approach of Fokker and Planck is unable to provide [13].

3.2 Noise models. To obtain the output covariance as a specific function of time, a definite time dependence must be assigned to the semi-invariants of the noise. An immense variety of types is naturally possible; three models felt to be important are treated here. In these models the entire time dependence is contained in the envelope of the pulses, although in general, variations in amplitude, shape, and phase of the individual pulses will also affect the time dependence. Variations in the amplitude or shape appear in the semi-invariants in the same way as the variations of pulse envelope, so that no loss in generality is to be expected if we attribute the overall time variation to a single cause rather than to a mixture of several. This is not true for the phase variation, however, since phase changes are associated with the high-frequency part of the semi-invariants, and therefore affect the noise distributions in a manner different from the amplitudes or pulse envelopes. Most noise sources produce pulses of constant phase, so that further refinements are unnecessary; however, the commonest exception, moving clutter in a radar system, is certainly an important one.*

Pertinent data on the three models are summarized in Table 1. The exponential pulses represent impulses passed through a single-tuned circuit. Although a single selective element is not an accurate model of the tuned stages of a receiver, this type of time dependence is important because it is necessary (and sufficient) if the process is to be Markoffian in the limit of increasing density. A representative physical case would be that of atmospheric static interference in a crystal video receiver.

The pass-band of an actual receiver is an involved function of the number of stages and the coupling networks between stages. As an abstraction from the details associated with any particular *i-f* strip, we shall take a pass-band of gaussian frequency response which preserves the essential features of a pass-band while remaining analytically tractable. Admittedly, a gaussian pass-band is not physically realizable, but it is a good approximation to the magnitude-frequency curve of actual amplifiers (if not to the phase frequency curve), and possesses the important virtue of simplicity**. Impulses passing through this *i-f* amplifier will become gaussian pulses, as in the second model of Table 1. "Gaussian" here refers to the pulse shape as a function of time and not to the statistics of the pulses.

The third type chosen is that of a square pulse envelope of finite duration. This exemplifies noise whose values are independent when separated by a sufficiently long, but finite time. The model fits the sonar and radar clutter problem when relative motion between the transceiver and scatterers is slight.

Notice that, in all these cases, the semi-invariants can be expressed as functions, in fact powers, of the input covariance function†. Accordingly, the time enters the output covariance only implicitly through the input covariance function.

3.3 Derivation of the output correlation function. The total covariance function of the output is the ensemble average of the product of the outputs as two times separated

*The alterations in the form of the covariance function which are necessary to take account of phase variations in time can be found in [11].

**H. Wallman has shown that the pass-band of cascaded networks whose individual step function responses have no overshoot tends toward the gaussian [17]. Furthermore, it seems likely that one can extend these results to any cascaded network except those tuned for a Butterworth response.

†The linear model is one extreme of possible time behavior in that its semi-invariants of all orders decrease no faster than the correlation function. One may conjecture that no $\Lambda_{22}(t)$, see Table 1, can decrease faster than the square of the covariance function, in which case the two other models represent the opposite extreme; however, we have been unable to prove this.

TABLE 1

Noise Models.

Pulse envelope $h(x)$	$\begin{cases} e^{-x}, & x > 0 \\ 0, & x < 0 \end{cases}$	e^{-x^2}	$\begin{cases} 1, & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$
Normalized covariance function $r_0(t)$	$e^{-\beta\mid t\mid}$	$e^{-\beta^2 t^2/2}$	$\begin{cases} 1 - \beta \mid t \mid, & \beta \mid t \mid < 1 \\ 0, & \beta \mid t \mid > 1 \end{cases}$
Normalized semi-invariants			
$\dfrac{\Lambda_{40}}{3!\psi^2}$	$1/4\gamma$	$1/4\gamma(\pi)^{1/2}$	$1/4\gamma$
$\dfrac{\Lambda_{31}(t) + \Lambda_{13}(t)}{3!\psi^2}$	$\dfrac{r_0 + r_0^3}{4\gamma}$	$r_0^{3/2}/2\gamma(\pi)^{1/2}$	$r_0/2\gamma$
$\dfrac{\Lambda_{22}(t)}{3!\psi^2}$	$r_0^2/4\gamma$	$r_0^2/4\gamma(\pi)^{1/2}$	$r_0/4\gamma$
$\dfrac{2^{(m+n)/2}}{(m+n)!}\dfrac{\Lambda_{mn}(t)}{\psi^{(m+n)/2}}$	$\dfrac{\gamma(2/\gamma)^{(m+n)/2}}{(m+n)\Gamma^2\left(\frac{m+n}{2}+1\right)}\begin{cases} r_0^m, & t > 0 \\ r_0^n, & t < 0 \end{cases}$	$\dfrac{\gamma(\pi)^{1/2}\left[\frac{1}{\gamma}\left(\frac{2}{\pi}\right)^{1/2}\right]^{(m+n)/2} r_0^{2mn/(m+n)}}{(m+n)^{1/2}\Gamma^2\left(\frac{m+n}{2}+1\right)}$	$\dfrac{r_0}{\gamma^{[(m+n)/2]-1}\Gamma^2\left(\frac{m+n}{2}+1\right)}$

by an interval t. The average can be expressed as a suitable integral over the characteristic function. We have [14, 18]

$$R_T(t) = \langle g(V_1)g(V_2)\rangle = \frac{1}{4\pi^2}\iint_C d\xi_1\, d\xi_2 f(i\xi_1)f(i\xi_2)\langle \exp(iV_1\xi_1 + iV_2\xi_2)\rangle, \tag{24}$$

where the average defines the characteristic function and

$$f(i\xi) = \int_{-\infty}^{\infty} g(V)e^{-iV\xi}\, dV, \qquad \operatorname{Im} \xi < 0 \tag{25}$$

in which $g(V)$ is zero for V less than some V_0, and of no greater than exponential order at infinity, and **C** is a straight line in the complex ξ-plane parallel to the real axis and lying below the singularities of $f(i\xi)$.

Since the input signal ensemble is the sum (see (21)) of sine wave and noise ensembles, which are independent of each other, the characteristic function of Eq. (24) becomes the product of the separate characteristic functions of the signal and of the noise. The results of Sec. II enable us to obtain the covariance function for non-gaussian noise, if we can obtain the corresponding one for non-stationary gaussian noise. Since the non-stationary gaussian case is only a very slight extension of previous work, it can easily be solved by methods briefly described below.

The characteristic function of the sine wave signal $A_0 \cos(\omega_0 t + \varphi)$ is [14]

$$F_2(\xi_1, \xi_2; t)_S = J_0(A_0[\xi_1^2 + \xi_2^2 + 2\xi_1\xi_2 \cos\omega_0 t]^{1/2}) \tag{26}$$

and the characteristic function of the noise is given by the exponential of Eq. (17). In order to find the zonal structure of the output, the two characteristic functions can be expanded in Fourier series, giving for the characteristic function of Eq. (24)

$$F_2(\xi_1, \xi_2; t) = \exp\left(-\frac{\psi_1\xi_1^2 + \psi_2\xi_2^2}{2}\right)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n}{2}\epsilon_m\epsilon_n J_n(A_0\xi_1)J_n(A_0\xi_2)$$
$$[I_{m+n}(\psi r_0\xi_1\xi_2)\cos(m\omega_0 t + n\alpha) + I_{|m-n|}(\psi r_0\xi_1\xi_2)\cos(m\omega_0 t - n\alpha)], \tag{27}$$

where the Neumann factor ϵ_m equals one for $m = 0$ and is two for all other values.

For the output of the receiver, only the low-frequency zone, $m = 0$, is required*. If now the modified Bessel function is expanded in a power series, the double integral of Eq. (24) separates into two single integrals, which have been evaluated previously [19], so that the covariance function of the low-frequency zone is

$$R_G(t) = \frac{\beta^2\Gamma^2(\nu+1)}{4\Gamma^2(\nu/2+1)}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\epsilon_n(\psi^2 r_0^2/\psi_1\psi_2)^{k+n/2}}{k!\,(k+n)!\,n!^2}\left(\frac{\psi_1\psi_2}{4}\right)^{\nu/2}(p_1p_2)^{n/2}\left(-\frac{\nu}{2}\right)_{n+k}^2 \cos n\alpha$$
$$\cdot {}_1F_1(-\nu/2 + n + k; n+1; -p_1)\, {}_1F_1(-\nu/2 + n + k; n+1; -p_2), \tag{28}$$

where p_i is the signal-to-noise power ratio, $A_0/2\psi_i$, ${}_1F_1$ is Kummer's form of the confluent hypergeometric function, and $(a)_n = \Gamma(a+n)/\Gamma(a)$.

3.4 Strong signals. When the signal is strong, i.e. p_i is large, Eq. (28) can be simplified by using the asymptotic expansion of the confluent hypergeometric function,

*Note that, if the ordinary asymptotic form of the characteristic function of the noise is used, all the harmonics must be retained until the cross-terms between the frequency-dependent semi-invariants have been taken into account.

which gives

$$R_G(t) \simeq \frac{\beta^2\Gamma^2(\nu+1)}{4\Gamma^4(\nu/2+1)}\left(\frac{A_0^2}{4}\right)^\nu \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\epsilon_n}{k!\,(k+n)!}(-\nu/2)_k^2(-\nu/2)_{n+k}^2(r_0/p)^{n+2k}\cos n\alpha$$

$$\cdot{}_2F_0(-\nu/2+n+k,\ -\nu/2+k;\ 1/p_1)\cdot{}_2F_0(-\nu/2+n+k,\ -\nu/2+k;\ 1/p_2), \tag{29}$$

where the formal hypergeometric function ${}_2F_0$ is defined as

$${}_2F_0(a,\ b;\ x) = \sum_{n=0}^{\infty}(a)_n(b)_n\frac{x^n}{n!}. \tag{30}$$

The asymptotic expansions used here are valid only when $p \gg |-\nu/2 + n + k|$. However, it can be shown [20] that the remainder contributed by the various possible series, e.g. when $n \gg k \gg p$, $k \gg p \doteq n$, etc., is of higher order than the remainder indicated by taking a finite part of Eq. (31), provided that $p > |-\nu/2 + n + k|$ for all terms which are included.

The differentiation of the gaussian result to obtain the non-gaussian one is readily accomplished with the help of the formula

$$\frac{d}{dx}\,{}_2F_0(a,\ b;\ cx) = abc\ {}_2F_0(a+1,\ b+1;\ cx). \tag{31}$$

When the leading terms of the power series are collected together, we find finally that the high and low density cases give the same result, viz:

$$\begin{aligned} R(t) \simeq \frac{\beta^2\Gamma^2(\nu+1)}{4\Gamma^4(\nu/2+1)}\left(p\,\frac{\psi}{2}\right)^\nu \Big\{ & 1 + 2(\nu/2)^2\,\frac{1+r_0}{p} + \frac{(\nu/2)^2}{p^2}\,[(\nu/2)^2(1+r_0^2) \\ & + (\nu/2-1)^2(1+r_0^2\cos 2\alpha) + 4(\nu/2)(\nu/2-1)r_0\cos\alpha] \\ & + \frac{(\nu/2)^2}{3p^2\psi^2}\,[(\nu/2-1)^2\Lambda_{40} + 2\nu/2(\nu/2-1)[\Lambda_{31}(t)+\Lambda_{13}(t)] \\ & + [2(\nu/2)^2 + (\nu/2-1)^2]\Lambda_{22}(t)] + 0(p^{-3}\gamma^{-2})\Big\}. \end{aligned} \tag{32}$$

The order of the remainder is different as the noise density becomes large or small. The terms in p^{-2} are of order p^{-2} and $p^{-2}\gamma^{-1}$; the next set of terms are of order p^{-3}, $p^{-3}\gamma^{-1}$, and $p^{-3}\gamma^{-2}$. For nearly gaussian noise, γ is large so that Eq. (32) can be used if p alone is large. For low-density noise, however, γp must be large as well as p.

In physical terms, γp is the ratio of signal power to the noise power in a single noise pulse. When the noise density is small, there is no noise at all for an appreciable part of the time; thus the requirement on γp means that the signal must be relatively strong while the noise pulse is present, which is here a more stringent requirement than that it be strong on the average.

When the signal is strong, there is, as usual, a modulation suppression effect [14] in that the leading term is the one for signal alone, with noise entering as a first-order correction. Here, however, there is an additional suppression effect on the statistics of the noise. As Eq. (32) shows, the non-gaussian nature of the noise is first discernible in the second-order terms.

3.5 Weak signals. The results when the signal is weak are more complicated because the low-density and high-density cases can no longer be described by the same formula.

If the confluent hypergeometric functions of Eq. (20) are expressed in series, one sum can be obtained in closed form, leaving a triple power series in the p_i , viz;

$$R_G(t) = \frac{\beta^2\Gamma^2(\nu+1)}{4\Gamma^2(\nu/2+1)}\left(\frac{\psi_1\psi_2}{4}\right)^{\nu/2}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\epsilon_n(-1)^{k+l}}{k!\,l!\,n!}\frac{p_1^{k+n/2}}{(k+n)!}$$
$$\cdot\frac{p_2^{l+n/2}}{(l+n)!}\left(\frac{\psi^2 r_0^{2n/2}}{\psi_1\psi_2}\right)\cos n\alpha\ \mathfrak{F}\left\{0, 0, 0; \frac{\psi^2 r_0^2}{\psi_1\psi_2}\right\}, \tag{33}$$

where

$$\mathfrak{F}\{a, b, c; x\} = (-\nu/2)_{k+n+a}(-\nu/2)_{l+n+a}$$
$$\cdot {}_2F_1(-\nu/2+k+n+a, -\nu/2+l+n+b; n+1+c; x). \tag{34}$$

Applying the differential operator of Eq. (17) one finds for the covariance function for nearly gaussian noise

$$R(t) = \frac{\beta^2\Gamma^2(\nu+1)}{4\Gamma^2(\nu/2+1)}\left(\frac{\psi}{2}\right)^{\nu}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\epsilon_n(-1)^{k+l}\cos n\alpha}{n!\,(n+k)!\,(n+l)!}$$
$$\cdot\frac{r_0^n p^{n+k+l}}{k!\,l!}\left\{\mathfrak{F}\{0, 0, 0; r_0^2\} + \frac{1}{3\psi^2}G(t)\right\}, \tag{35}$$

where

$$G(t) = \Lambda_{40}\mathfrak{F}\{2, 0, 0; r_0^2\} + [\Lambda_{31}(t) + \Lambda_{13}(t)]\left[\frac{n}{r_0}\mathfrak{F}\{1, 0, 0; r_0^2\}\right.$$
$$\left. + \frac{2r_0}{n+1}\mathfrak{F}\{2, 1, 1; r_0^2\}\right] + \Lambda_{22}(t)\left[\frac{n(n-1)}{2r_0^2}\mathfrak{F}\{0, 0, 0; r_0^2\}\right.$$
$$\left. + \frac{3n+2}{n+1}\mathfrak{F}\{1, 1, 0; r_0^2\} + \frac{r_0^2}{(n+1)^2(n+2)}\mathfrak{F}\{2, 2, 2; r_0^2\}\right]. \tag{36}$$

The hypergeometric functions reduce to polynomials when ν is an even integer and to complete elliptic integrals when ν is an odd integer.

When the noise density is small, one must use the characteristic function Eq. (20), which involves much the same sort of derivative as does the high-density case just treated. In addition, however, there are the time intervals in which no noise pulse or one noise pulse occurs. If there is no noise present, the correlation function is that for signal alone, which was found as the first term of Eq. (32).

If a single noise pulse alone is present, the exact form of the distributions must be used. On the other hand, when the signal is present at the same time as a single noise pulse, the covariance function is less sensitive to the statistics of the noise, and it becomes possible to approximate the noise distribution in the usual way. For a single pulse the covariance function is most easily found by not carrying out the average until the end. Thus the characteristic-function of the noise can be written as

$$F_1(\xi_1, \xi_2; t) = \langle J_0([v_1^2\xi_1^2 + v_2^2\xi_2^2 + 2\xi_1\xi_2 v_1 v_2\cos(\omega_0 t - \alpha)]^{1/2})\rangle, \tag{37}$$

where the average is over the parameters of $v(t_0 - \tau)$, an individual noise pulse.

This characteristic function can be expanded in a Fourier series by the usual Bessel function addition theorem, whereupon the double integral for the low frequency zone

splits into the product of single integrals each of which is of Weber's type [21], so that the correlation function is

$$\frac{\beta^2\Gamma^2(\nu+1)}{4\Gamma^4(\nu/2+1)}\left(\frac{\psi}{2}\right)^\nu \frac{\langle v_1^\nu v_2^\nu\rangle}{\gamma^\nu\langle v^2\rangle}. \tag{38}$$

If now the differential operator of Eq. (20) is applied to Eq. (33), we obtain finally for a weak signal and low-density noise

$$\begin{aligned} R(t) = \frac{\beta^2\Gamma^2(\nu+1)}{4\Gamma^2(\nu/2+1)}\left(\frac{\psi}{2}\right)^\nu &\left\{\frac{p^\nu e^{-\gamma}}{\Gamma^2(\nu/2+1)}\right. \\ &+ \frac{e^{-\gamma}}{\gamma^{\nu-1}\Gamma^2(\nu/2+1)}\frac{\langle v_1^\nu v_2^\nu\rangle}{\langle v^2\rangle^\nu} \\ &+ \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\epsilon_n(-1)^{k+l}r_0^n\cos n\alpha}{n!\,(n+k)!\,k!\,(n+l)!\,l!}p^{k+l+n} \\ &\cdot[\gamma^{-\nu+n+k+l}\varphi(\gamma,\nu-n-k-l)\mathfrak{F}\{0,0,0;r_0^2\} \\ &+\frac{1}{3\psi^2}\gamma^{1-\nu+n+k+l}\varphi(\gamma,\nu-1-n-k-l)G(t) \\ &-\tfrac{1}{2}(\nu-k-l-n)(\nu-k-l-k-1) \\ &\left.\cdot\gamma^{-\nu+n+k+l}\varphi(\gamma,\nu-n-k-l)\mathfrak{F}\{0,0,0;r_0^2\}]\right\}, \end{aligned} \tag{39}$$

where $G(t)$ is defined in Eq. (36) and

$$\phi(\gamma,\beta) \equiv \sum_{k=1}^{\infty}\frac{\gamma^k}{k!}k^\beta e^{-\gamma}\ ; \tag{40}$$

φ is a polynomial in γ and $\exp(-\gamma)$ for positive integral β, and an iterated integral of the exponential integral for negative integral β; for other values, a closed form is unavailable.

3.6 The quadratic detector. The output correlation function after a half-wave quadratic detector is particularly simple. When ν equals 2, the output voltage is the same, except for a scale factor, as that from a full-wave squaring device, because the noise and signal distributions are symmetric about $v = 0$. The covariance function of all zones is thus a fourth-degree moment of the input. The covariance function of the low-frequency zone is then

$$R(t) = \frac{\beta^2\psi^2}{4}\left[1 + r_0^2 + \frac{2\Lambda_{22}(t)}{3\psi^2} + 2p(1+r_0) + p^2\right], \tag{41}$$

where $p = A_0^2/2\psi$, the input signal-to-noise ratio, and $r_0 = r_0(t)$. The equation is exact for all values of signal-to-noise ratio or noise density. Finding the output covariance function as a moment of the input voltage is by far the simplest approach; however, the method may be applied only when ν is an even integer.

Results for the three noise models listed above in Sec. 3.2 are given in Table 2 and illustrated for the exponential and linear models in Figs. 2, 3, and 4. The figures show that the square-law rectifier reduces the amount of correlation compared to that present

TABLE 2

Covariance functions for the quadratic detector.

1. Exponential model:

$$r_0 = e^{-\beta|t|}$$

$$R(t) = \frac{\beta^2\psi^2}{4}\left[(1+p)^2 + 2pr_0 + \left(1+\frac{1}{\gamma}\right)r_0^2\right]$$

2. Gaussian model:

$$r_0 = \exp\left(-\frac{\beta^2 t^2}{2}\right)$$

$$R(t) = \frac{\beta^2\psi^2}{4}\left[(1+p)^2 + 2pr_0 + \left(1+\frac{1}{\gamma(\pi)^{1/2}}\right)r_0^2\right]$$

3. Linear model:

$$r_0 = \begin{cases} 1 - \beta\,|\,t\,|, & \beta\,|\,t\,| < 1 \\ 0, & \beta\,|\,t\,| > 1 \end{cases}$$

$$R(t) = \frac{\beta^2\psi^2}{4}\left[(1+p)^2 + \left(2p+\frac{1}{\gamma}\right)r_0 + r_0^2\right]$$

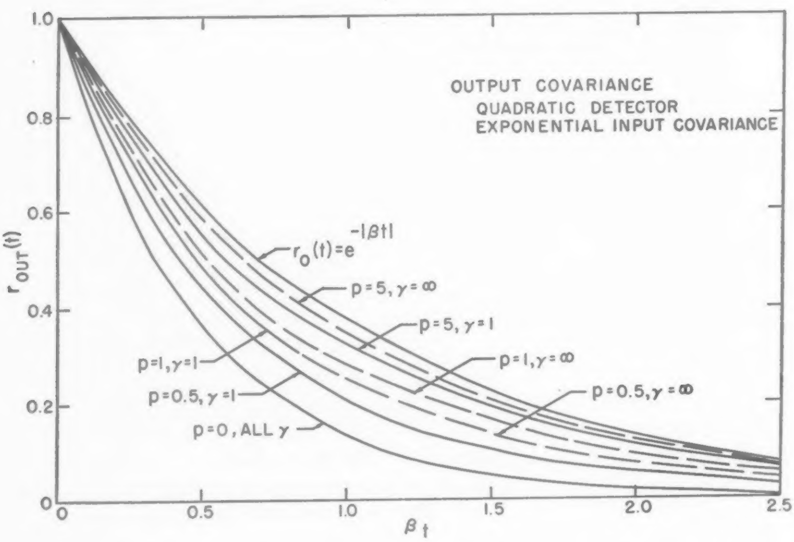

FIG. 2. Output covariance for the quadratic detector (exponential input covariance).

in the input for all values of p and γ. As p increases, the curves for all values of noise density increase too. This is because the signal $\times$ noise intermodulation terms become progressively more important as compared to the noise $\times$ noise products and, since the $s \times n$ contribution is more correlated*, $r_{out}(t)$ is increased. For large p a non-gaussian

*The expression "more correlated" is used throughout the paper to express the fact that one normalized output correlation function is greater than another for the same value of t when both are derived from the same input correlation function.

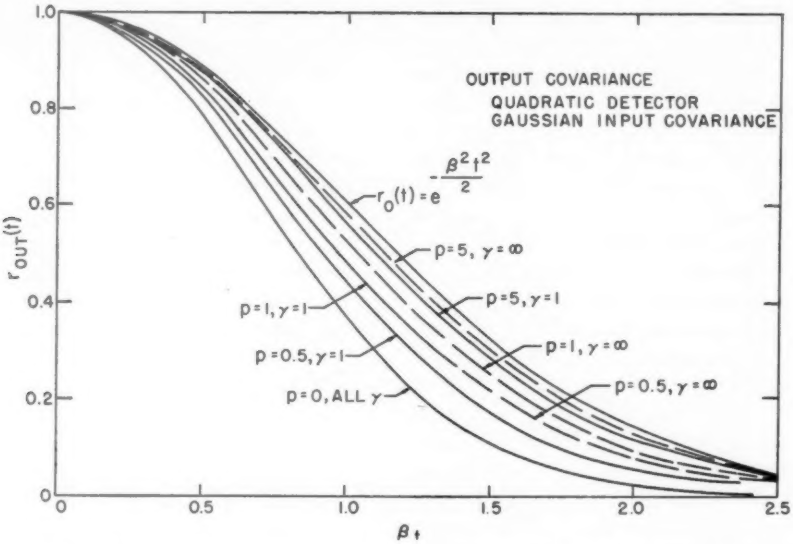

FIG. 3. Output covariance for the quadratic detector (gaussian input covariance).

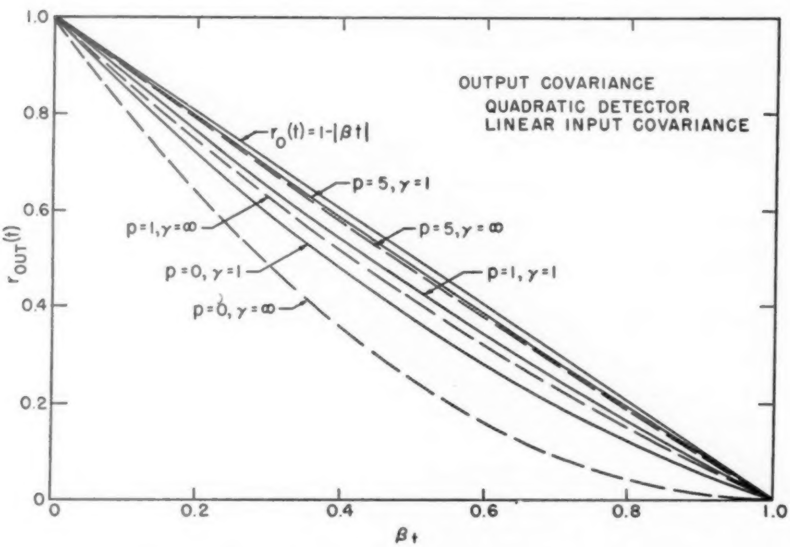

FIG. 4. Output covariance for the quadratic detector (linear input covariance).

suppression effect enters, so that the difference between gaussian and non-gaussian noise is less than at low signal levels.

The effect of the non-normal nature of the noise here is to reduce the output correlation compared to the corresponding gaussian noise for the exponential and gaussian

noise models and to increase it for the linear model. Because of the simplicity of the expression for the covariance function, it is easy to see why this is so. For gaussian input noise, the quadratic detector output has one distorted noise term (r_0^2 , for $n \times n$ modulation). For non-normal noise, an additional term enters, containing the semi-invariant $\Lambda_{22}(t)$. This term adds to the "scrambled" noise for the exponential and gaussian models, and to the undistorted noise for the linear model. Clearly, the output correlation is usually decreased by non-gaussian statistics since, for nearly all variations, $\Lambda_{22}(t)$ will decrease faster than $r_0(t)$. The linear model is the extreme case in which this is not so.

3.7 The linear detector. Obtaining the correlation function for a half-wave linear detector involves all the considerations, if not quite the full algebraic complexity, which attend the general case. No elementary method of obtaining the output covariance function is possible for a half-wave linear detector. However, when the general νth-law results of preceding sections are specialized to $\nu = 1$, considerable simplification occurs. The hypergeometric functions of the general result reduce to complete elliptic integrals and the results for large and small density become much alike. Table 3 lists the correlation functions for the three noise models. The three cases of a strong signal, a weak signal with high-density noise, and a weak signal with low-density noise are listed separately. The expressions in the table are the leading terms of series in the noise density γ and signal-to-noise ratio p. The nearly gaussian and low-density results hold over a considerable range of variation of γ, because the numerical coefficients of higher terms are small. The strong signal formulas of the table are reasonably accurate down to values of p of about 2. The weak signal correlation function, on the other hand, will hold only in a restricted range, for p no more than about 1/2, because only the first two terms of the ascending p series have been used.

TABLE 3

Covariance functions for the linear detector[*].

a. Strong signal, $p > 1$

1. Exponential model: $r_0 = \exp(-\beta \mid t \mid)$

$$R(t) = \frac{2\beta^2\psi p}{\pi^2}\left\{1 + \frac{1 + r_0}{2p} + \frac{1 - r_0}{8p^2}\left[1 - r_0 + \frac{1}{4\gamma}(1 - r_0 + 2r_0^2)\right]\right.$$
$$\left. + \frac{1 - r_0^2}{16p^3}\left[1 - r_0 + \frac{1}{4\gamma}(5 - 5r_0 + 2r_0^2)\right] + 0(p^{-4}, \gamma^{-2})\right\}$$

2. Gaussian model: $r_0 = e - \dfrac{\beta^2 t^2}{2}$

$$R(t) = \frac{2\beta^2\psi p}{\pi^2}\left\{1 + \frac{1 + r_0}{2p} + \frac{1}{8p^2}\left[(1 - r_0)^2 + \frac{1}{4\gamma(\pi)^{1/2}}(1 - 4r_0^{3/2} + 3r_0^2)\right]\right.$$
$$\left. + \frac{1}{16p^3}\left[(1 - r_0^2)(1 - r_0) + \frac{1 + r_0}{4\gamma(\pi)^{1/2}}(5 - 8r_0 - 4r_0^{3/2} + 7r_0^2)\right\} + 0(p^{-4}, \gamma^{-2})$$

---

*Here $K(r_0)$ and $E(r_0)$ are complete elliptic integrals of the first and second kind, respectively, and $B(r_0) = E + (1 - r_0^2)K$, $r_0^4C(r_0) = (2 - r_0^2)K - 2E$, $r_0^2D(r_0) = K - E$. (See Jahnke and Emde, "Tables of Functions," Dover, New York, 1945, pp. 73-80.) Also $\phi(\gamma, -1)$ is obtained from Eq. (40).

3. Linear model:
$$r_0 = \begin{cases} 1 - \beta \mid t \mid, & \beta \mid t \mid < 1 \\ 0, & \beta \mid t \mid > 1 \end{cases}$$

$$R(t) = \frac{2\beta^2 \psi p}{\pi^2} \left\{ 1 + \frac{1 + r_0}{2p} + \frac{1 - r_0}{8p^2} \left[1 - r_0 + \frac{1}{4\gamma} \right] \right.$$
$$\left. + \frac{1 - r_0}{16p^3} \left[1 - r_0^2 + \frac{1}{4\gamma}(5 - 3r_0) \right] + 0(p^{-4}, \gamma^{-2}) \right.$$

b. Weak signal, low-density noise, $p < 1, \gamma < 1$

1. Exponential model:
$$r_0 = \exp(-\beta \mid t \mid)$$

$$R(t) = \frac{\beta^2 \psi}{\pi^2} \left\{ [E(r_0) + r_0^2 B(r_0)][1 - e^{-\gamma}] + \frac{4}{\pi} \gamma e^{-\gamma} r_0 \right.$$
$$- (1 - r_0^2) \frac{1 - (1 + \gamma)e^{-\gamma}}{8\gamma} [2E(r_0) - K(r_0)] + p\left[\frac{4}{\pi} e^{-\gamma} \right.$$
$$+ \{1 - e^{-\gamma}\}\{E(r_0) + r_0 B(r_0)\}$$
$$\left.\left. + \frac{\phi(\gamma, -1)}{8} \{(1 - 2r_0^2)E(r_0) + r_0 B(r_0)\} \right] + 0(p^2) \right\}$$

2. Gaussian model:
$$r_0 = \exp\left(- \frac{\beta^2 t^2}{2} \right)$$

$$R(t) = \frac{\beta^2 \psi}{\pi^2} \left\{ [E(r_0) + r_0^2 B(r_0)][1 - e^{-\gamma}] + \frac{4}{\pi} \gamma e^{-\gamma} r_0 \right.$$
$$+ \frac{1 - (1 + \gamma)e^{-\gamma}}{8\gamma(\pi)^{1/2}} [4r_0^2 D(r_0) - (1 - r_0^2)K(r_0) - 4r_0^{5/2} D(r_0)]$$
$$+ p\left[\frac{4}{\pi} e^{-\gamma} + \{1 - e^{-\gamma}\}\{E(r_0) + r_0 B(r_0)\} \right.$$
$$\left. + \frac{\phi(\gamma, -1)}{8(\pi)^{1/2}(1 - r_0^2)} \{(1 - 5r_0^2 + 4r_0^{5/2})E(r_0) + r_0(3 - 4r_0^{1/2} + r_0^2)B(r_0)\} \right]$$
$$\left. + 0(p^2) \right\}$$

3. Linear model:
$$r_0 = \begin{cases} 1 - \beta \mid t \mid & \beta \mid t \mid < 1 \\ 0, & \beta \mid t \mid > 1 \end{cases}$$

$$R(t) = \frac{\beta^2 \psi}{\pi^2} \left\{ [E(r_0) + r_0 B(r_0)][1 - e^{-\gamma}] + \frac{4}{\pi} \gamma e^{-\gamma} r_0 \right.$$
$$- (1 - r_0) \frac{1 - (1 + \gamma)e^{-\gamma}}{8\gamma} [K(r_0) - 2r_0 D(r_0)]$$
$$+ p\left[\frac{4}{\pi} e^{-\gamma} + \{1 - e^{-\gamma}\}\{E(r_0) + r_0 B(r_0)\} \right.$$
$$\left. + (1 - r_0) \frac{\phi(\gamma, -1)}{8} \left\{ \frac{3B(r_0)}{1 + r_0} - D(r_0) \right\} \right]$$
$$\left. + 0(p^2) \right\} \phi(\gamma, -1) = e^{-\gamma} \sum_{n=1}^{\infty} \frac{\gamma^n}{n!n}$$

c. Weak signal, nearly gaussian noise, $p < 1, \gamma > 1$

1. Exponential model: $r_0 = \exp(-\beta \mid t \mid)$

$$R(t) = \frac{\beta^2\psi}{\pi^2}\left\{E(r_0) + r_0^2B(r_0) - \left(\frac{1-r_0^2}{8\gamma}\right)[2E(r_0) - K(r_0)] + p\left[E(r_0) + r_0B(r_0) + \frac{1}{8\gamma}\{(1-2r_0^2)E(r_0) + r_0B(r_0)\}\right] + 0(p^2, \gamma^{-2})\right\}$$

2. Gaussian model: $r_0 = \exp\left(-\frac{\beta^2t^2}{2}\right)$

$$R(t) = \frac{\beta^2\psi}{\pi^2}\left\{E(r_0) + r_0^2B(r_0) + \frac{1}{8\gamma(\pi)^{1/2}}[4r_0^2D(r_0) - (1-r_0^2)K(r_0) - 4r_0^{5/2}D(r_0)] + p\left[E(r_0) + r_0B(r_0) + \frac{1}{8\gamma(1-r_0^2)(\pi)^{1/2}}\{(1-5r_0^2+4r_0^{5/2})E(r_0) + r_0(3 - 4r_0^{1/2} + r_0^2)B(r_0)\}\right] + 0(p^2, \gamma^{-2})\right\}$$

3. Linear model: $r_0 = \begin{cases} 1 - \beta \mid t \mid, & \beta \mid t \mid < 1 \\ 0, & \beta \mid t \mid > 1 \end{cases}$

$$R(t) = \frac{\beta^2\psi}{\pi^2}\left\{E(r_0) + r_0^2B(r_0) - \frac{1-r_0}{8\gamma}[K(r_0) - 2r_0D(r_0)] + p\left[E(r_0) + r_0B(r_0) + \frac{1-r_0}{8\gamma}\left\{\frac{3}{1+r_0}B(r_0) - D(r_0)\right\}\right] + 0(p^2, \gamma^{-2})\right\}$$

The normalized output correlation function is shown in Figs. 5, 6, and 7 with $r_{out}(t)$ for gaussian noise of the same input power shown in dashed lines. One may note, as for the quadratic detector, that increases in p cause the amount of correlation in the output to increase and cause the difference between gaussian and non-gaussian noise to decrease; however, the results have several features which are distinct from those for the quadratic detector. For example, as the noise density decreases, the leading term in the low-density series, $\langle v'(t_0)v'(t_0 + t)\rangle$, which for the linear detector is $r_0(t)$, becomes increasingly important, so that the output has more correlation than the output for a gaussian input noise for all noise models. This effect will occur as the noise density becomes sufficiently small for fixed signal power, even though the (average) signal-to-noise ratio is large. When γp becomes small, the weak signal series must be used.

Determination of the extent to which the nearly normal and low-density results together cover the range of γ is an important question which the covariance functions for the linear detector partially answer. Apparently the shape of the noise pulses has considerable effect. For gaussian pulses, the two forms of the correlation function are in close agreement (too close to be resolved in the figure); for the exponential and linear pulses, the agreement is not so complete. The figures tend to show that when the noise density is about 1, the two forms of the covariance function will be only roughly equal, but that no gross differences occur.

3.8 Effects of non-gaussian statistics in general. The linear and quadratic detectors are the most important, so that the results applicable to them have been presented in

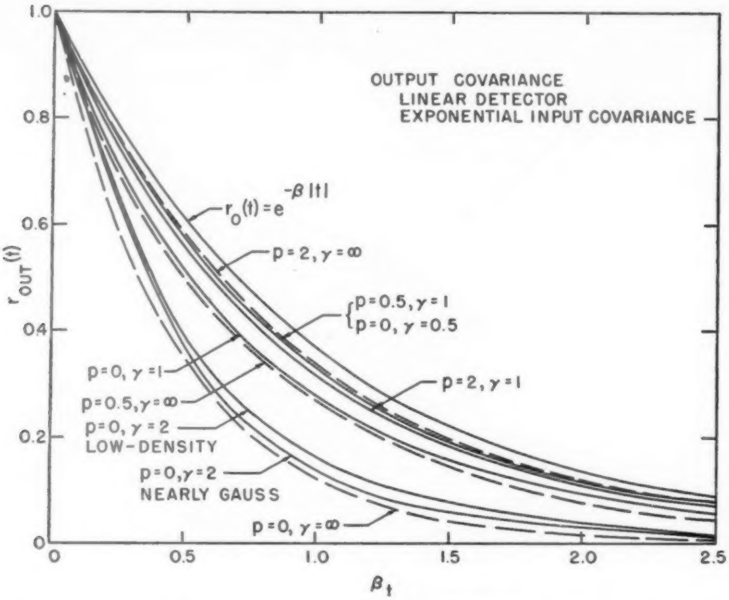

FIG. 5. Output covariance for the linear detector (exponential input covariance).

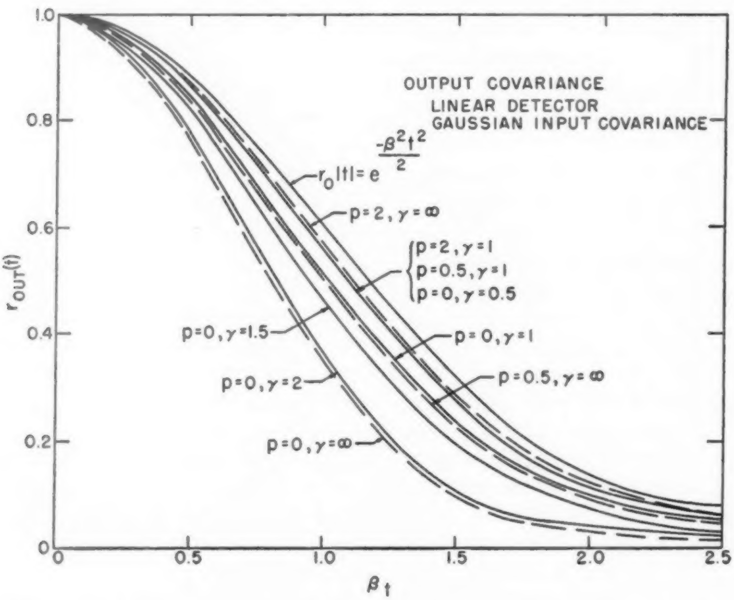

FIG. 6. Output covariance for the linear detector (gaussian input covariance).

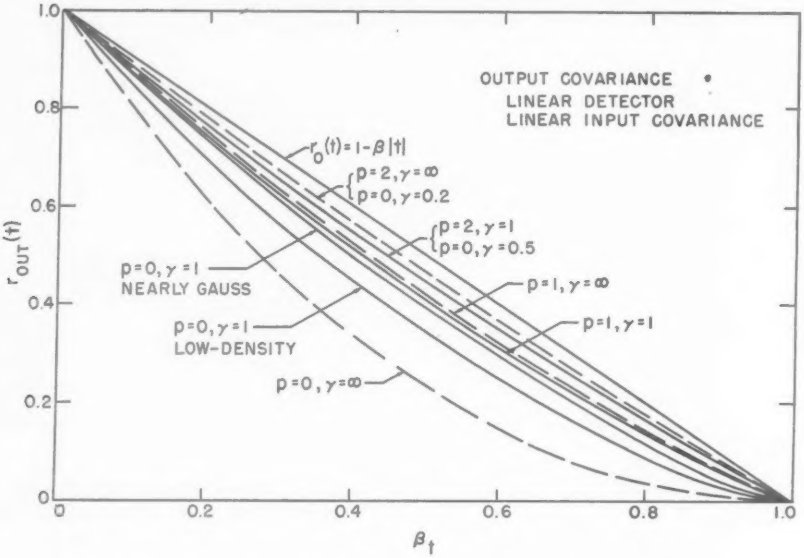

FIG. 7. Output covariance for the linear detector (linear input covariance).

considerable detail. A qualitative look at the output correlation function for arbitrary ν will indicate the over-all features of these results and indicate the place of the linear and quadratic detectors in the general scheme.

Comparison of the covariance functions of gaussian and non-gaussian noise can be carried out for the weak signal cases by comparing the leading terms of the expansions of the hypergeometric functions. Separate expansions must be used when r_0^2 is near zero and when r_0 is near one. Since in both limiting regions, the same results emerge from the analysis, we may expect with a reasonable degree of confidence that the results hold for all values of $r_0^2(t)$, and hence all t. If the signal is strong, all the functions involved are elementary, so that the comparison is much easier to make. In both cases, the algebra is at once straightforward, lengthy, and not especially illuminating, so that it will not be reproduced here [11, 22]. Figure 8 shows the results of such a comparison. In the shaded region the non-gaussian and gaussian correlation functions intersect each other, so that neither can be said to be larger, although for most values of t the non-gaussian output is less correlated. No scale has been indicated on the vertical axis because the position of the boundaries depends on the density of the noise. Deviations from the gaussian result are small for large signals and large for small signals.

As mentioned in Sec. 3.2, the noise models chosen here represent nearly the extremes of variation, so that we may expect the same general type of behavior for other shapes of noise pulses. Unless the semi-invariants of high degree have the same time structure [e.g. in the linear model, they all equal $r_0(t)$], the type of behavior shown by the gaussian or exponential noise model should be representative.

3.9 The output spectrum. The variation of the output power as a function of frequency is an important statistic of the output. It can be obtained as the cosine trans-

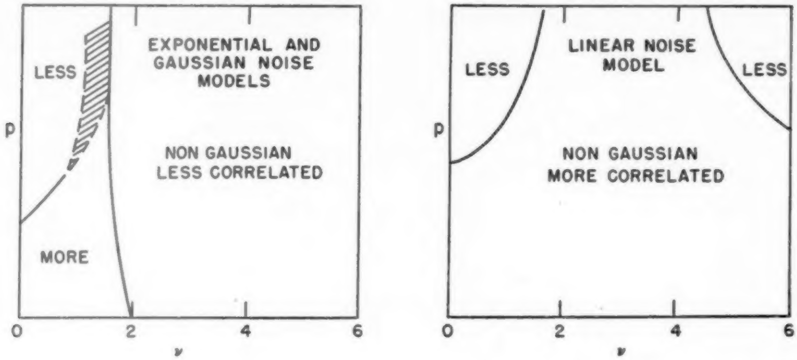

FIG. 8. General effect of non-gaussian statistics.

form of the covariance function,

$$W(f) = 4 \int_0^\infty R(t) \cos \omega t \, dt$$

$$R(t) = \int_0^\infty W(f) \cos \omega t \, df. \tag{42}$$

The covariance function is the quantity found directly from the analysis of the detector and therefore has been presented in greater detail than the spectrum will be. In general, the only tractable expressions for the covariance function are approximate, and the transform even of these can be taken only approximately. The covariance function and spectrum are equivalent in principle; and the propagation of error in taking cosine transforms can be minimized by giving results primarily in terms of the covariance function.

The general behavior of the spectrum can be inferred from the reciprocal spreading property of Fourier transforms, i.e. the more a function is concentrated in one domain, the more it is dispersed in the transform domain [23].

A more quantitative measure of the spectrum can be found by the use of equivalent rectangular bandwidths. The low-frequency output spectrum has a maximum at zero frequency and decreases smoothly as the frequency increases. One can construct a spectrum having a constant value over a finite bandwidth equal to the zero frequency value of the actual spectrum and the value zero elsewhere, such that the total power in the actual and rectangular spectra is the same for both. Then the total bandwidth of the rectangular spectrum is the equivalent rectangular bandwidth of the actual spectrum. It is customary not to include the δ-function at zero frequency representing d-c power in the actual spectrum.

Explicitly, one has

$$f_0 = \frac{\int_0^\infty W_0(f) \, df}{W_0(0)}, \tag{43}$$

where $W_0(f)$ is the spectrum minus the d-c contribution. In terms of the correlation function, the bandwidth is

$$f_0 = \frac{R(0) - R(\infty)}{4 \int_0^\infty [R(t) - R(\infty)]\, dt} = \left[4 \int_0^\infty r_{\text{out}}(t)\, dt \right]^{-1}. \tag{44}$$

Table 4 gives the ratio of the output bandwidth to the input bandwidth for all the curves of Figs. 2 to 7. The entries in the table give a quantitative measure of the difference

TABLE 4

*Spectral bandwidths**.

Quadratic Detector

Exponential covariance function			Gaussian covariance function			Linear covariance function		
$p = 0$,	all γ	2.00	$p = 0$,	all γ	1.41	$p = 0$,	$\gamma = \infty$	1.50
$p = .5$,	$\gamma = \infty$	1.33	$p = .5$,	$\gamma = \infty$	1.17	$p = 0$,	$\gamma = 1$	1.20
$p = .5$,	$\gamma = 1$	1.50	$p = .5$,	$\gamma = 1$	1.28	$p = 1$,	$\gamma = \infty$	1.12
$p = 1$,	$\gamma = \infty$	1.20	$p = 1$,	$\gamma = \infty$	1.11	$p = 1$,	$\gamma = 1$	1.09
$p = 1$,	$\gamma = 1$	1.33	$p = 1$,	$\gamma = 1$	1.19	$p = 2$,	$\gamma = \infty$	1.07
$p = 5$,	$\gamma = \infty$	1.05	$p = 5$,	$\gamma = \infty$	1.03	$p = 2$,	$\gamma = 1$	1.06
$p = 5$,	$\gamma = 1$	1.09	$p = 5$,	$\gamma = 1$	1.05	$p = 5$,	$\gamma = \infty$	1.03+
						$p = 5$,	$\gamma = 1$	1.03−

Linear Detector

Exponential covariance function			Gaussian covariance function			Linear covariance function		
$p = 0$,	$\gamma = \infty$	2.25	$p = 0$,	$\gamma = \infty$	1.48	$p = 0$,	$\gamma = \infty$	1.59
$p = 0$,	$\gamma = 2$NG	1.86	$p = 0$,	$\gamma = 2$	1.42	$p = 0$,	$\gamma = 1$LD	1.28
$p = 0$,	$\gamma =$ LD	1.55	$p = 0$,	$\gamma = 1.5$	1.19	$p = 0$,	$\gamma = 1$NG	1.18
$p = 0$,	$\gamma = 1$	1.25	$p = 0$,	$\gamma = 1$	1.15	$p = 0$,	$\gamma = .5$	1.08
$p = 0$,	$\gamma = .5$	1.11	$p = 0$,	$\gamma = .5$	1.09	$p = 0$,	$\gamma = .2$	1.05
$p = .5$,	$\gamma = \infty$	1.44	$p = .5$,	$\gamma = \infty$	1.12	$p = 1$,	$\gamma = \infty$	1.13
$p = .5$,	$\gamma = 1$	1.37	$p = .5$,	$\gamma = 1$	1.08	$p = 1$,	$\gamma = 1$	1.14
$p = 2$,	$\gamma = \infty$	1.12	$p = 2$,	$\gamma = \infty$	1.06	$p = 2$,	$\gamma = \infty$	1.05
$p = 2$,	$\gamma = 1$	1.10	$p = 2$,	$\gamma = 1$	1.08	$p = 2$,	$\gamma = 1$	1.08

*The entries in the table are the ratio of the bandwidth at the detector output to the bandwidth at the detector input.
NG means "nearly gaussian"; LD means "low density"

between gaussian and non-gaussian noise. Since the bandwidth is larger as the covariance function is smaller, the results are just opposite to those of the previous section.

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Appendix A

Nearly normal distributions in terms of parametric derivatives. An expansion of a nearly normal distribution in a series of derivatives with respect to the second moments is useful when the noise ensemble has a phase uniformly distributed, independent of the envelope. As has been pointed out in Sec. 2.3, nearly normal narrow-band noise waves have such a structure.

The terms of fixed degree, $M = m + n$, in the moment expansion of the second-order characteristic function (Eq. (3)) are

$$\begin{aligned}\sum_{\substack{m,n \\ m+n=M}} \frac{\langle R_1^m R_2^n \rangle}{m!\,n!} \langle \cos^m (\omega_0 t_0 - \theta_1) \cos^n (\omega_0[t_0 + t] - \theta_2)\rangle (i\xi_1)^m (i\xi_2)^n \\ = \frac{1}{M!} \langle [i\xi_1 R_1 \cos (\omega_0 t_0 - \theta_1) + i\xi_2 R_2 \cos (\omega_0[t_0 + t] - \theta_2)]^M \rangle.\end{aligned} \tag{A.1}$$

The average over $\omega_0 t_0 - \theta_1$ can be carried out as a contour integral around the unit circle, after the usual substitution, $z = \exp\{i(\omega_0 t_0 - \theta_1)\}$. There is one singularity inside the contour, a pole of order $M + 1$ at the origin; the integral is zero if M is odd, and, when M is even, gives

$$\frac{\langle \cos^M \theta \rangle}{M!} \langle [(i\xi_1 R_1)^2 + (i\xi_2 R_2^2) + 2(i\xi_1 R_1)(i\xi_2 R_2) \cos(\omega_0 t - \theta)]^{M/2} \rangle. \tag{A.2}$$

For a non-random phase change, to which we here restrict ourselves, the expansion is found compactly through the formal use of tensor notation including the summation convention.

Let the vector $\{X_i\}$, $(i = 1, 2, 3)$ be defined as

$$\{X_i\} = \{R_1^2, R_1 R_2, R_2^2\}, \tag{A.3a}$$

and let

$$\{\Xi_i\} = \{\xi_1^2, 2\xi_1\xi_2 \cos(\omega_0 t - \alpha), \xi_2^2\}, \tag{A.3b}$$

$$\{D_i\} = \left\{\frac{\partial}{\partial \psi_1}, \frac{\partial}{\partial \psi r_0}, \frac{\partial}{\partial \psi_2}\right\}, \tag{A.3c}$$

so that the expression (A.2) becomes, with $M = 2p$,

$$\frac{(-1)^p}{(2p)!} \langle \cos^{2p} \theta \rangle \langle (X_i \Xi_i)^p \rangle = \frac{(-1)^p}{(2_p)!} \langle \cos^{2p} \theta \rangle \langle X_{n_1} \cdots X_{n_p} \rangle \Xi_{n_1} \cdots \Xi_{n_p}. \tag{A.4}$$

The first factors of (A.4) can be recognized as an envelope factor for the moments [from Eq. (12)] so that we can write

$$\frac{(-1)^p}{(2p)!} \mathfrak{M}_{n_1 \cdots n_p} \Xi_{n_1} \cdots \Xi_{n_p}, \tag{A.5}$$

where

$$\mathfrak{M}_{n_1 \cdots n_p} = \langle \cos^{2p} \theta \rangle \langle R_1^{3p - \Sigma n_i} R_2^{\Sigma n_i - p} \rangle = M_{3p - \Sigma n_i, \Sigma n_i - p}.$$

Thus the moment expansion of the characteristic function, which in general is a double series [see Eq. (3)], can be written as a single series when the noise is narrow band,

$$F_2(\xi_1, \xi_2; t) = \sum_{p=0}^{\infty} \frac{(-1)^p}{(2p)!} \mathfrak{M}_{n_1 \cdots n_p} \Xi_{n_1} \cdots \Xi_{n_p}. \tag{A.6}$$

By comparing terms with the semi-invariant expansion of the characteristic function, one finds that, similarly,

$$F_2(\xi_1, \xi_2; t) = \exp\left\{\sum_{q=1}^{\infty} \frac{(-1)^q}{(2q)!} L_{n_1 \cdots n_q} \Xi_{n_1} \cdots \Xi_{n_q}\right\}, \tag{A.7}$$

where

$$L_{n_1 \cdots n_q} = \Lambda_{3q - \Sigma n_i, \Sigma n_i - q},$$

and, in particular, $L_{ij} = \Lambda_{4-i-j, i+j-2}$, $L_{ijk} = \Lambda_{9-i-j-k, i+j+k-3}$, $(i, j, k = 1, 2, 3)$.

If the exponential in Eq. (A.7) is expanded in its power series except for its first term,

it is readily seen that each Ξ_i may be replaced by a $2D_i$, giving the desired expansion whose initial terms are

$$
\begin{aligned}
F_2(\xi_1, \xi_2; t) = \Big\{1 &+ \frac{2^2}{4!} L_{ij} D_i D_j + \frac{2^3}{6!} L_{ijk} D_i D_j D_k \\
&+ \tfrac{1}{2} \frac{2^4}{4!^2} L_{ij} L_{kl} D_i D_j D_k D_l + 0(\gamma^{-3})\Big\} \exp\{-\tfrac{1}{2} L_i \Xi_i\}\Big|_{\psi_1 = \psi = \psi_2}, \qquad \text{(A.8)}
\end{aligned}
$$

from which the manner of formation of further terms is evident.

SOLUTIONS OF THE HELMHOLTZ EQUATION FOR A CLASS OF NON-SEPARABLE CYLINDRICAL AND ROTATIONAL COORDINATE SYSTEMS*

BY

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Abstract. In cylindrical and rotational coordinate systems, one of the variables can be separated out of the Helmholtz equation, leaving a second order partial differential equation in two variables. For a class of the coordinate systems, this equation is reducible to a recurrence set of ordinary differential equations in one variable, which are solvable by ordinary methods.

1. Introduction. The usual method of solving the Helmholtz equation

$$\nabla^2\psi + k^2\psi = 0 \tag{1}$$

in three dimensions is the method of separation of variables, in which the equation is separated into three ordinary differential equations, each of which can be solved. However, it is shown by Eisenhart [1] that the method of separation of variables in euclidean space is applicable to only eleven coordinate systems, generated by confocal quadrics or their degenerate forms. Exact solutions of the Helmholtz equation have thus hitherto been limited to separable coordinate systems.

In this article a method is presented for solving (1) for a class of non-separable cylindrical and rotational coordinate systems.

2. Cylindrical and rotational coordinate systems. The definitions of cylindrical and rotational coordinate systems are those given by P. Moon and D. Spencer [2]. Let $z = F(w)$, where $z = x_1 + iy_1$, and F is any analytic function. Separating real and imaginary parts, one obtains

$$x_1 = \xi_1(u, v), \qquad y_1 = \xi_2(u, v). \tag{2}$$

The curves u = constant, and v = constant give rise to an orthogonal family of curves in the z-plane. For each function F, a cylindrical coordinate system and one or two rotational coordinate systems can be formed.

The cylindrical system (u_1, u_2, u_3) is given by the equations

$$\begin{aligned} x &= \xi_1(u_1, u_2), \\ y &= \xi_2(u_1, u_2), \\ z &= u_3, \end{aligned} \tag{3}$$

and the rotational coordinate systems $(u_1, u_2, u_3 = \phi)$ by

$$\begin{aligned} x &= \xi_1(u_1, u_2)\cos\phi, \\ y &= \xi_1(u_1, u_2)\sin\phi, \\ z &= \xi_2(u_1, u_2), \end{aligned} \tag{4}$$

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and

$$
\begin{aligned}
x &= \xi_2(u_1, u_2) \cos \phi, \\
y &= \xi_2(u_1, u_2) \sin \phi, \\
z &= \xi_1(u_1, u_2).
\end{aligned} \tag{5}
$$

System (4) is found by rotation about the y_1-axis and system (5) about the x_1-axis. The z-axis is then taken as the axis of rotation. The metric coefficients for the cylindrical coordinate systems have the property that

$$
h_1^2 = h_2^2 = \left(\frac{\partial \xi_1}{\partial u_1}\right)^2 + \left(\frac{\partial \xi_1}{\partial u_2}\right)^2, \qquad h_3 = 1 \tag{6}
$$

and for the rotational coordinate system (4)

$$
\begin{aligned}
h_1^2 &= h_2^2 = \left(\frac{\partial \xi_1}{\partial u_1}\right)^2 + \left(\frac{\partial \xi_1}{\partial u_2}\right)^2 \\
h_3 &= \xi_1(u_1, u_2)
\end{aligned} \tag{7}
$$

and rotational coordinate system (5)

$$
\begin{aligned}
h_1^2 &= h_2^2 = \left(\frac{\partial \xi_2}{\partial u_1}\right)^2 + \left(\frac{\partial \xi_2}{\partial u_2}\right)^2 \\
h_3 &= \xi_2(u_1, u_2).
\end{aligned} \tag{8}
$$

Also since F is an analytic function $\xi_i(u_1, u_2)$ has the property

$$
\begin{gathered}
\frac{\partial^2 \xi_i}{\partial u_1^2} + \frac{\partial^2 \xi_i}{\partial u_2^2} = 0, \\
\frac{\partial \xi_1}{\partial u_1} = \frac{\partial \xi_2}{\partial u_2}, \qquad \frac{\partial \xi_1}{\partial u_2} = -\frac{\partial \xi_2}{\partial u_1}.
\end{gathered} \tag{9}
$$

Now rotational and cylindrical coordinates have the property that one of the variables (ϕ and u_3 respectively) can be separated out of (1) leaving a second order partial differential equation in the two variables u_1 and u_2. We shall show that there exists a class of coordinate systems for which this equation can be reduced to a recurrence set of ordinary differential equations in one variable.

3. Solutions of the Helmholtz equation for rotational coordinate systems.

Theorem I. If for a rotational coordinate system there is a $u_i (i \neq 3)$ such that $h_3(u_1, u_2)$ has the property that

$$
\frac{\partial h_3}{\partial u_i} = \sum_{s=N_1}^{M_1} f_s(u_i)[h_3(u_1, u_2)]^s, \tag{10}
$$

where N_1, M_1 are integers, and $N_1 \leq M_1$ and if the metric coefficient h_1^2 has the form

$$
h_1^2 = \sum_{s=N_2}^{M_2} c_s(u_i)[h_3(u_1, u_2)]^s, \tag{11}
$$

where N_2, M_2 are integers and $N_2 \leq M_2$ then

$$
\psi(u_1, u_2, \phi) = e^{i\mu\phi} \sum_r a_r(u_i)[h_3(u_1, u_2)]^r \tag{12}
$$

is a solution of the equation $\nabla^2\psi + k^2\psi = 0$, where $a_r(u_i)$ satisfies the set of recurrence relations

$$\frac{d^2a_r}{du_i^2} + \sum_{q=r+1-N_1}^{r+1-M_1} \frac{da_q}{du_i}(2q+1)f_{r+1-q}(u_i) + \sum_{q=r+2-N_2}^{r+2-M_2}(q^2-\mu^2) \cdot a_q(u_i)c_{r+2-q}(u_i) + k^2\sum_{q=r-N_2}^{r-M_2} a_q(u_i)c_{r-q}(u_i) = 0. \tag{13}$$

Proof. The operator ∇^2 in rotational coordinate systems has the form

$$\frac{1}{h_1^2h_3}\left\{\frac{\partial}{\partial u_1}\left(h_3\frac{\partial}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(h_3\frac{\partial}{\partial u_2}\right) + \frac{h_1^2}{h_3}\frac{\partial^2}{\partial\phi^2}\right\}.$$

Hence using the expression for ψ given by (12), Eq. (1) reduces to

$$e^{i\mu\phi}\left\{\frac{1}{h_1^2h_3}\left[\frac{\partial}{\partial u_1}\left(h_3\frac{\partial}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(h_3\frac{\partial}{\partial u_2}\right)\right] - \frac{\mu^2}{h_3^2} + k^2\right\}\cdot\sum a_r(u_i)(h_3)^r = 0. \tag{14}$$

Multiplying (14) by $h_1^2\exp(-i\mu\phi)$ one obtains the following

$$\sum_r\left\{\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} + \frac{1}{h_3}\frac{\partial h_3}{\partial u_1}\frac{\partial}{\partial u_1} + \frac{1}{h_3}\frac{\partial h_3}{\partial u_2}\frac{\partial}{\partial u_2} - \mu^2\frac{h_1^2}{h_3^2} + k^2h_1^2\right\}\cdot a_r(u_i)(h_3)^r = 0,$$

$$\sum_r\left\{(h_3)^r\frac{d^2a_r}{du_i^2} + 2\frac{da_r}{du_i}r(h_3)^{r-1}\frac{\partial h_3}{\partial u_i} + \left(k^2h_1^2 - \mu^2\frac{h_1^2}{h_3^2}\right)a_r(h_3)^r + r(h_3)^{r-2}a_r\left[(r-1)\left(\frac{\partial h_3}{\partial u_1}\right)^2 + (r-1)\left(\frac{\partial h_3}{\partial u_2}\right)^2 + h_3\frac{\partial^2h_3}{\partial u_1^2} + h_3\frac{\partial^2h_3}{\partial u_2^2}\right] + r(h_3)^{r-1}\frac{a_r}{h_3}\left[\frac{\partial h_3}{\partial u_1}\frac{\partial h_3}{\partial u_1} + \frac{\partial h_3}{\partial u_2}\frac{\partial h_3}{\partial u_2}\right] + \frac{(h_3)^r}{h_3}\frac{\partial h_3}{\partial u_i}\frac{da_r}{du_i}\right\} = 0.$$

Using relations (7), (8) and (9) one obtains

$$\sum_r\left\{(h_3)^r\frac{d^2a_r}{du_i^2} + (2r+1)\frac{da_r}{du_i}(h_3)^{r-1}\frac{\partial h_3}{\partial u_i} + r^2a_r(h_3)^{r-2}h_1^2 + \left(k^2h_1^2 - \mu^2\frac{h_1^2}{h_3^2}\right)a_r(h_3)^r\right\} = 0. \tag{15}$$

Use relations (10) and (11), and arrange (15) in power series in $h_3(u_1, u_2)$ to obtain

$$\sum_r(h_3)^r\left\{\frac{d^2a_r}{du_i^2} + \sum_{q=r+1-N_1}^{r+1-M_1}\frac{da_q}{du_i}(2q+1)f_{r+1-q}(u_i) + \sum_{q=r+2-N2}^{r+2-M_2}(q^2-\mu^2)a_q(u_i)c_{r+2-q}(u_i) + \sum_{q=r-N_2}^{r-M_2}k^2a_q(u_i)c_{r-q}(u_i)\right\} = 0.$$

Hence equate coefficients of $(h_3)^r$ to zero. A recurrence set of ordinary differential equations is then obtained relating the functions $a_r(u_i)$

$$\frac{d^2a_r}{du_i^2} + \sum_{q=r+1-N_1}^{r+1-M_1}\frac{da_q}{du_i}(2q+1)f_{r+1-q}(u_i) + k^2\sum_{q=r-N_2}^{r-M_2}a_q(u_i)c_{r-q}(u_i) + \sum_{q=r+2-N_2}^{r+2-M_2}(q^2-\mu^2)a_q(u_i)c_{r+2-q}(u_i) = 0. \tag{16}$$

Thus provided that the ordinary differential equations given by (16) can be solved for the $a_r(u_i)$, the Helmholtz equation has the solution

$$\psi(u_1, u_2, \phi) = e^{i\mu\phi} \sum_r a_r(u_i)[h_3(u_1, u_2)]^r. \tag{17}$$

Here the summation over r represents a power series in the function (h_3).

There are two cases of practical importance, (i) lower termination of series (17) when $M_2 \geq N_2 \geq 2$ and $M_1 \geq N_1 \geq 1$ and (ii) upper termination when $N_2 \leq M_2 \leq 0$ and $N_1 \leq M_1 \leq 1$.

For the case of lower termination the differential equation (16) is an inhomogeneous equation, the homogeneous part involving a_r and the inhomogeneous involving $a_{r-1}, a_{r-2}, \cdots$ i.e. terms a_n such that $n < r$. Thus, if $a_{r-1}, a_{r-2}, \cdots$ are known then a_r can be found. Now there is some number p such that $a_{p-1} \equiv a_{p-2} \equiv a_{p-3} \equiv 0$. Hence the inhomogeneous portion of (16) for $r = p$, is zero. Thus a_p is a solution of the homogeneous equation and can be found. Since Eq. (16) for $r = p + 1$ involves only a_p and a_{p+1} and a_p is now known, then a_{p+1} can be found. Hence if $a_p, a_{p+1}, \cdots, a_{r-1}$ are known, then a_r can be found.

Hence one has a solution of the form

$$\left.\begin{aligned} \psi_p^\mu(u_1, u_2, \phi) &= e^{i\mu\phi} \sum_{r=p,p+1}^{\infty} a_r(u_i)(h_3)^r \\ &= e^{i\mu\phi}(h_3)^p \sum_{N=0}^{\infty} a_{p+N}(h_3)^N. \end{aligned}\right\} \tag{18}$$

Similarly it can be shown for the case of upper termination that if q is such that $a_{q+1} \equiv a_{q+2} \equiv a_{q+3} \equiv 0$, the expression (17) becomes

$$\left.\begin{aligned} \psi_q^\mu(u_1, u_2, \phi) &= e^{i\mu\phi} \sum_{-\infty}^{r=q,q-1} a_r(u_i)(h_3)^r \\ &= e^{i\mu\phi}(h_3)^q \sum_{N=0}^{\infty} a_{-N+q}(u_i)(h_3)^{-N}. \end{aligned}\right\} \tag{19}$$

The numbers p and q are determined from boundary conditions. In addition to the solutions of the form (17) there may exist other solutions described in the following theorem.

Theorem II. If for a rotational coordinate system there is a $u_i (i \neq 3)$ such that $h_3(u_1, u_2)$ and h_1^2 have the properties given by (10) and (11) respectively, and if there is a function $B(u_j)$ where $i \neq j \neq 3$; such that

$$\frac{dB}{du_j}\frac{\partial h_3}{\partial u_j} = B(u_j) \sum_{s=N_3}^{M_3} d_s(u_i)(h_3)^s, \tag{20}$$

where N_3 and M_3 are integers and $N_3 \leq M_3$ and

$$\frac{d^2B}{du_j^2} = B(u_j) \sum_{s=N_4}^{M_4} e_s(u_i)(h_3)^s \tag{21}$$

where N_4 and M_4 are integers and $N_4 \leq M_4$ then

$$\psi(u_1, u_2, \phi) = e^{i\mu\phi}B(u_j) \sum_r b_r(u_i)[h_3(u_1, u_2)]^r \tag{22}$$

is a solution of the Helmholtz equation $\nabla^2\psi + k^2\psi = 0$ where $b_r(u_i)$ must satisfy the set of recurrence relations

$$\frac{d^2b_r}{du_i^2} + \sum_{q=r+1-N_1}^{r+1-M_1} \frac{db_q}{du_i}(2q+1)f_{r+1-q} + \sum_{q=r+2-N_2}^{r+2-M_2} (q^2-\mu^2)b_q c_{r+2-q}$$
$$+ k^2 \sum_{q=r-N_2}^{r-M_2} b_q c_{r-q} + \sum_{r-N_4}^{r-M_4} b_q e_{r-q} + \sum_{q=r-M_3+1}^{r-M_3+1} b_q(2q+1)\, d_{r+1-q} = 0.$$

The proof is similar to that of Theorem 1. As before, there are two cases of practical importance for (22); upper termination and lower termination of the series.

When $M_2 \geq N_2 \geq 2$, $M_1 \geq N_1 \geq 1$, $M_4 \geq N_4 \geq 0$ and $N_3 \geq N_3 \geq 1$ there are solutions of the Helmholtz equation of the form

$$\psi^{\mu}_{p'}(u_1, u_2, \phi) = e^{i\mu\phi}B(u_j) \sum_{r=p'}^{\infty} b_r(u_i)[h_3(u_1, u_2)]^r \tag{23}$$

and when $N_2 \leq M_2 \leq 0$, $N_1 \leq M_1 \leq 1$, $N_4 \leq M_4 \leq 0$ and $N_3 \leq M_3 \leq 1$ there are solutions of the form

$$\psi^{\mu}_{q'}(u_1, u_2, \phi) = e^{i\mu\phi}B(u_j) \sum_{-\infty}^{r=q'} b_r(u_i)[h_3(u_1, u_2)]^r. \tag{24}$$

The second type of solutions given by (23) and (24) are essential if h_3 is an even function of the variable u_j. Then the first type of solution given by (17) are even functions of u_j, and if odd solutions in the variable u_j are required, then this $B(u_j)$ is chosen such that $B(u_j)$ is an odd function of u_j.

4. Solutions of the Helmholtz equation for cylindrical coordinate systems. In a similar manner as for the rotational case, solutions of the Helmholtz equation can be obtained and are stated in the following theorem (where to simplify presentation the two cases corresponding to Theorem I and Theorem II are combined):

Theorem III. If for a cylindrical coordinate system there is a $u_i (i \neq 3)$ and $\xi_k(u_1, u_2)$ $(k \neq 3)$ such that

$$\frac{\partial \xi_k}{\partial u_i} = \sum_{s=N_1}^{M_1} b_s(u_i)[\xi_k(u_1, u_2)]^s, \tag{25}$$

where N_1 and M_1 are integers and $N_1 \leq M_1$ and if the metric coefficient h_1 has the property that

$$h_1^2 = \sum_{s=N_2}^{M_2} c_s(u_i)[\xi_k(u_1, u_2)]^s, \tag{26}$$

where N_2 and M_2 are integers and $N_2 \leq M_2$ and there exists a function $B(u_j)$ where $i \neq j \neq 3$ such that

$$\frac{d^2B(u_j)}{du_j^2} = B(u_j) \sum_{s=N_4}^{M_4} e_s(u_i)[\xi_k(u_1, u_2)]^s \tag{27}$$

and

$$\frac{dB}{du_j}\frac{\partial \xi_k}{\partial u_j} = B(u_j) \sum_{s=N_3}^{M_3} d_s(u_i)[\xi_k(u_1, u_2)]^s, \tag{28}$$

where N_3, M_3, N_4 and M_4 are integers and $N_3 \leq M_3$, $N_4 \leq M_4$. Then

$$\psi(u_1, u_2, u_3) = e^{i\mu u_3}B(u_j) \sum_r a_r(u_i)[\xi_k(u_1, u_2)]^r \tag{29}$$

is a solution of the Helmholtz equation $\nabla^2\psi + k^2\psi = 0$ and $a_r(u_i)$ satisfies the set of recurrence relations

$$\begin{aligned} \frac{d^2a_r}{du_i^2} + \sum_{q=r+1-N_1}^{r+1-M_1} \frac{da_q}{du_i}(2q)b_{r+1-q}(u_i) + \sum_{q=r+2-N_2}^{r+2-M_2} q(q-1)a_q(u_i)c_{r+2-q}(u_i) \\ + (k^2 - \mu^2)\sum_{q=r-N_3}^{r-M_3} a_q(u_i)c_{r-q}(u_i) + \sum_{q=r-N_4}^{r-M_4} a_q(u_i)e_{r-q}(u_i) \\ + \sum_{q=r+1-N_3}^{r+1-M_3} 2qa_q(u_i)\, d_{r-q+1} = 0. \end{aligned} \tag{30}$$

A special case of the expression given by (29) corresponding to (17) for rotational coordinates, is given when $B(u_j) = 1$ and $d_s(u_i) = 0$ for $s = N_3, N_3 + 1, \cdots, M_3$ and $e_s(u_i) = 0$ for $s = N_4, N_4 + 1, \cdots, M_4$. There are two cases of practical importance:

(i) Lower termination, $M_2 \geq N_2 \geq 2$, $M_1 \geq N_1 \geq 1$, $M_4 \geq N_4 \geq 0$, $M_3 \geq N_3 \geq 1$

(ii) Upper termination, $N_2 \leq M_2 \leq 0$, $N_1 \leq M_1 \leq 1$, $N_4 \leq M_4 \leq 0$, $N_3 \leq M_3 \leq 1$.

For lower termination the expression given by (29) reduces to

$$\psi_p^\mu(u_1, u_2, u_3) = e^{i\mu u_3}B(u_j) \sum_{r=p}^{\infty} a_r(u_i)[\xi_k(u_1, u_2)]^r \tag{31}$$

and for upper termination

$$\psi_q^\mu(u_1, u_2, u_3) = e^{i\mu u_3}B(u_j) \sum_{-\infty}^{r=q} a_r(u_i)[\xi_k(u_1, u_2)]^r. \tag{32}$$

5. Particular coordinate systems. The question of practicability of the above method of solving Eq. (1) for the particular non-separable coordinate systems under consideration is answered by the application of the method to the toroidal coordinate system. The author has obtained a complete set of solutions of Eq. (1) in toroidal coordinates, which satisfy the radiation condition and possess a ring singularity [3]. Besides toroidal coordinates, Eq. (1) can be solved for other well-known coordinate systems, as is shown in Table 1. The solutions given there, are independent of prescribed boundary conditions. The expressions given are in terms of power series of $f(u_i)\ h_3(u_1, u_2)$ and $g(u_i)\ \xi_k(u_1, u_2)$ instead of $h_3(u_1, u_2)$ and $\xi_k(u_1, u_2)$ according to whether the system is rotational or cylindrical. An appropriate choice of $f(u_i)$ or $g(u_i)$ simplifies the differential equations (13) and (30), transforming the homogeneous part into a recognizable form.

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TABLE 1

Coordinate system	*Series solution of* $\nabla^2\psi + k^2\psi = 0$	*Set of recurrence equations relating the coefficients*	*Solutions of the corresponding homogeneous equation*
Bipolar (ξ, η, z) $x = \dfrac{d \sinh \xi}{\cosh \xi - \cos \eta}$ $y = \dfrac{d \sin \eta}{\cosh \xi - \cos \eta}$ $z = z$	$\psi_e(s, \eta, z) = e^{i\mu z} \sum_{r=\tau}^{\infty} A_r(s)\,(s - \cos \eta)^{-r}$ where $s = \cosh \xi$	$(s^2 - 1)\dfrac{d^2 A_r}{ds^2} + s\dfrac{dA_r}{ds} - r^2 A_r + (k^2 - \mu^2)\, d^2 A_{r-2} - 2(r-1)\left[(s^2 - 1)\dfrac{dA_{r-1}}{ds} - (r-1)sA_{r-1}\right] = 0$	$(s \mp [s^2 - 1]^{\frac{1}{2}})^{r}$ $(s \mp [s^2 - 1]^{\frac{1}{2}})^{-r}$
	$\psi_0(s, \eta, z) = e^{i\mu z} \sin \eta \sum_{r=\tau'}^{\infty} B_r(s)\,(s - \cos \eta)^{-r}$	$(s^2 - 1)\dfrac{d^2 B_r}{ds^2} + s\dfrac{dB_r}{ds} - (r-1)^2 B_r + d^2(k^2 - \mu^2)B_{r-2} - 2(r-1)\left[(s^2 - 1)\dfrac{dB_{r-1}}{ds} - (r-2)sB_{r-1}\right] = 0$	$(s \mp [s^2 - 1]^{\frac{1}{2}})^{r-1}$ $(s \mp [s^2 - 1]^{\frac{1}{2}})^{-r+1}$
Bispherical (ξ, η, ϕ) $x = \dfrac{d \sin \eta \cos \phi}{\cosh \xi - \cos \eta}$	$\psi_e(s, \eta, \phi) = e^{i\mu\phi} \sum_{r=\tau}^{\infty} A_r(x)\,(s - \cos \eta)^{-r}$ where $x = \cos \eta$	$(1 - x^2)\dfrac{d^2 A_r}{dx^2} - 2x\dfrac{dA_r}{dx} + A_r\left[\dfrac{-\mu^2}{1 - x^2} + r(r+1)\right] + d^2k^2 A_{r-2} + (2r - 1)\left[(1 - x^2) \cdot \dfrac{dA_{r-1}}{dx} + (r-1)xA_{r-1}\right] = 0$	$P_r^{\mu}(x)$ $Q_r^{\mu}(x)$

$y = \dfrac{d \sin \eta \sin \phi}{\cosh \xi - \cos \eta}$ $z = \dfrac{d \sinh \xi}{\cosh \xi - \cos \eta}$	$\psi_o(s, \eta, \phi) = e^{i\mu\phi} \sinh \xi \sum_{r=r'}^{\infty} B_r(x)\,(s - \cos \eta)^{-r}$	$(1 - x^2)\dfrac{d^2 B_r}{dx^2} - 2x\dfrac{dB_r}{dx} + B_r\left[\dfrac{-\mu^2}{1 - x^2} + r(r-1)\right] + d^2k^2 B_{r-2} + (2r-1)\left[(1 - x^2) \cdot \dfrac{dB_{r-1}}{dx} + (r-2)xB_{r-1}\right] = 0$	$P^{\mu}_{r-1}(x)$ $Q^{\mu}_{r-1}(x)$
Toroidal (ξ, η, ϕ) $x = \dfrac{d \sinh \xi \cos \phi}{\cosh \xi - \cos \eta}$ $y = \dfrac{d \sinh \xi \sin \phi}{\cosh \xi - \cos \eta}$ $z = \dfrac{d \sin \eta}{\cosh \xi - \cos \eta}$	$\psi_e(s, \eta, \phi) = e^{i\mu\phi} \sum_{r=r}^{\infty} A_r(s)\,(s - \cos \eta)^{-r}$	$(s^2 - 1)\dfrac{d^2 A_r}{ds^2} + 2s\dfrac{dA_r}{ds} - A_r\left[\dfrac{+\mu^2}{(s^2 - 1)} + r(r+1)\right] + d^2k^2 A_{r-2} - (2r-1)\left[(s^2 - 1) \cdot \dfrac{dA_{r-1}}{ds} - s(r-1)A_{r-1}\right] = 0$	$P^{\mu}_{r}(s)$ $Q^{\mu}_{r}(s)$
	$\psi_o(s, \eta, \phi) = e^{i\mu\phi} \sin \eta \sum_{r=r'}^{\infty} B_r(s)\,(s - \cos \eta)^{-r}$	$(s^2 - 1)\dfrac{d^2 B_r}{ds^2} + 2s\dfrac{dB_r}{ds} - B_r\left[\dfrac{+\mu^2}{s^2 - 1} + r(r-1)\right] + d^2k^2 B_{r-2} - (2r-1)\left[(s^2 - 1) \cdot \dfrac{dB_{r-1}}{ds} - s(r-2)B_{r-1}\right] = 0$	$P^{\mu}_{r-1}(s)$ $Q^{\mu}_{r-1}(s)$

—NOTES—

ON LINEAR PERTURBATIONS*

By AUREL WINTNER (*The Johns Hopkins University*)

If $f(t)$ and $g(t)$ are continuous functions for large positive t, and if the solutions of the differential equation

$$x'' + f(t)x = 0 \tag{1}$$

are known, how "small" must be the difference $f(t) - g(t)$ (for large t) in order that the general solution of the differential equation

$$y'' + g(t)y = 0 \tag{2}$$

can be guaranteed to have the same asymptotic behavior as the general solution of (1)? The literature consulted answers this question only under assumptions which restrict (1) by certain conditions of stability.[1] No such assumptions are made in the following theorem (which, under stability assumptions, reduces to known results):

With reference to the coefficient function $f(t)$ and a pair $x = u(t)$, $x = v(t)$ of linearly independent solutions of (1), *let the coefficient function $g(t)$ of* (2) *satisfy the following condition*:

$$\int^{\infty} | f - g | (| u |^2 + | v |^2) \, dt < \infty. \tag{3}$$

Then every solution $y = y(t)$ of (2) *is of the form*

$$y(t) = c_1 u(t) + c_2 v(t) + o(| u(t) | + | v(t) |), \tag{4}$$

where c_1, c_2 are integration constants (which can be chosen arbitrarily). In addition, the asymptotic relation (4) *remains true on differentiation, that is*

$$y'(t) = c_1 u'(t) + c_2 v'(t) + o(| u'(t) | + | v'(t) |), \tag{5}$$

where $' = d/dt$. The o symbol in $h(t) = j(t) + o(| k(t)|)$ means that $k \neq 0$ for large t and that $(h - j)/k \to 0$ as $t \to \infty$.

The relation (4) reduces to $y \sim c_1 u + c_2 v$ as $t \to \infty$ if (1) is stable in the sense that $\limsup | x(t)| < \infty$, where $t \to \infty$, holds for all solutions of (1). In this case, condition (3) is certainly satisfied if

$$\int^{\infty} | f - g | \, dt < \infty. \tag{6}$$

But the converse conclusion cannot be made (not even $\limsup | x(t)| < \infty$ is assumed for $x = u$ and $x = v$), since, when $u(t)$ and $v(t)$ happen to be "small" (for large t), then condition (3) requires of the "size" of the perturbation $f - g$ substantially less than what is required by the standard assumption (6).

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[1]In this regard, see A. Wintner, Quart. Appl. Math. **13**, 192-195 (Secs. 2 and 5) (1955).

What is more, the general theorem applies without any stability restriction of the given problem (1). In fact, no matter what the coefficient and the general solution of (1) (that is, the function f and two, linearly independent, solutions u, v) may be, the perturbation $f - g$ of (2) on (1) can always be chosen so small as to satisfy condition (3), that is, so as to bring (2) within the range of the theorem.

The proof of the theorem will consist of two steps.

First, recourse will be had to an elementary lemma which goes back to Bôcher[2] and which, in the binary case at hand, runs as follows: If the coefficients of a homogeneous, linear differential system

$$p' = a_{11}(t)p + a_{12}(t)q, \qquad q' = a_{21}(t)p + a_{22}(t)q \tag{7}$$

are, for large positive t, continuous functions satisfying

$$\int^{\infty} | a_{ik}(t) | \, dt < \infty, \quad \text{where} \quad i = 1, 2 \quad \text{and} \quad k = 1, 2, \tag{8}$$

then, corresponding to any pair c_1, c_2 of integration constants, the system (7) possesses a unique solution (p, q) satisfying

$$p(t) \to c_1, \qquad q(t) \to c_2 \quad \text{as} \quad t \to \infty. \tag{9}$$

Next, the following rule of Lagrangian "variation of constants" (a rule which, being purely formal in nature, requires only the continuity of $f(t)$ and $g(t)$ on a t-interval) will be needed.[3] Let two solutions, $x = u$ and $x = v$, of (1) be so chosen that their Wronskian $u(t)v'(t) - v(t)u'(t)$ (which is always a non-vanishing constant) becomes the constant 1, and define, in terms of the difference of the coefficient functions of (1) and (2), a binary matrix function $\| a_{ik}(t) \|$ as follows:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = (f - g)\begin{pmatrix} -uv & -v^2 \\ u^2 & uv \end{pmatrix}. \tag{10}$$

Then $y(t)$ is a solution of (2) if and only if there belongs to it a solution (p, q) of the case (10) of (7) in such a way that

$$y(t) = u(t)p(t) + v(t)q(t) \tag{11}$$

becomes an identity.

In addition, this transformation of (2) into (1) is a "contact transformation," in the sense that the following differentiation rule holds for (11):

$$y'(t) = u'(t)p(t) + v'(t)q(t) \tag{12}$$

(in other words, $up' + vq'$ vanishes for all t)[4].

In order to combine this Lagrangian rule with Bôcher's lemma, note that the case (10) of the four conditions (8) is equivalent to the three conditions

$$\int^{\infty} | f - g | \, | w | \, dt < \infty, \quad \text{where} \quad w = u^2, uv, v^2,$$

[2]For references, and for certain refinements, see A. Wintner, Am. J. Math. **76**, 183-190 (1954).

[3]For a verification (and for a similar application) of this rule, see A. Wintner, Am. J. Math. **69**, 262-263 (1947).

[4]See formula (34) in the preceding reference[3].

and that, since $2 \mid uv \mid \leqq \mid u \mid^2 + \mid v \mid^2$, the latter three conditions are equivalent to the single condition (3). Accordingly, (3) assures the validity of the limit relations (9) for the general solution (p, q) of the case (10) of (7). But (11) reduces to (4), and (11) to (5), by virtue of (9).

This completes the proof of the theorem. Its assumption (3) is independent of the choice of the two, linearly independent, solutions $x = u(t)$, $x = v(t)$ of (1) which occur in (3). For, on the one hand, $u(t)$ and $v(t)$ cannot vanish at the same t and, on the other hand, the ratio of $\mid u^*(t) \mid^2 + \mid v^*(t) \mid^2$ to $\mid u(t) \mid^2 + \mid v(t) \mid^2$ stays between two positive constant bounds as $t \to \infty$. This is clear from the fact that $u^*(t) = au(t) + bv(t)$ and $v^*(t) = cu(t) + dv(t)$, where a, b, c, d are constants of non-vanishing determinant $ad - bc$.

A particular, but interesting, case of the theorem results if condition (3) is strengthened so as to make possible the elimination of the solution pair (u, v) occurring in (3). Such an elimination is made possible by using a well-known estimate, which was applied in more general forms by Liapounoff[5] and others (L. Schlesinger, G. D. Birkhoff, O. Perron)[6] and which, when particularized to the case of (1), states that

$$\mid x(t) \mid < \text{const.} \exp \{\tfrac{1}{2} \int^t \mid f(s) - 1 \mid ds\} \tag{13}$$

holds for every solution $x(t)$ of (1). What then results avoids the implicit hypothesis of the theorem, namely, that (1) has already been solved. In fact, the resulting corollary of the theorem can be formulated as follows.

If the coefficient function $g(t)$ of (2) is so "close" (for large t) to the coefficient function $f(t)$ of (1) that

$$\int^\infty \mid g(t) - f(t) \mid \exp \{\int^t \mid f(s) - 1 \mid ds\}\, dt < \infty, \tag{14}$$

then the asymptotic behavior of all solutions $y(t)$ of (2) and of their derivatives $y'(t)$ is given by (4) and (5), where c_1, c_2 is an arbitrary pair of integration constants and $u(t)$, $v(t)$ is a pair of linearly independent solutions $x(t)$ of (1).

In fact, if (13) is applied to $x = u$ and $x = v$, then (3) reduces to (14).

A LEBEDEV TRANSFORM AND THE "BAFFLE" PROBLEM*

By C. P. WELLS AND A. LEITNER (*Michigan State University*)

1. Introduction. This note is concerned with the application of the Lebedev transform to what we term the "baffle problem," i.e. the problem of sound radiated by a vibrating circular disk in an infinite rigid baffle. The solution of this problem is not new. It has been solved by Sommerfeld [1] in terms of cylindrical waves, using the Hankel transform, and by Bouwkamp [2] and others in terms of spheroidal waves, using series representation. However, the Lebedev transform offers an equally straightforward method, representing the radiation in terms of spherical waves, and moreover is in

[5]E. Picard, *Traité d'Analyse*, 3rd ed., vol. 3, 1928, p. 385.

[6]For the particular case (13) of the general theorem, see N. Levinson, Duke Math. J. 8, 2-3(1941).

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itself of considerable inherent interest. It has been applied by Kontorovich and Lebedev [3] and by Oberhettinger [4] to problems of diffraction by a wedge, and by Leitner and Wells [5] to the problem of a freely vibrating disk.

We use spherical coordinates r, θ, φ with the baffle in the plane $\theta = \pi/2$ and seek solutions, independent of φ, of

$$\nabla^2 u + k^2 u = 0, \tag{1}$$

where $k = 2\pi/\lambda$, λ = wave length, and $u = u(r, \theta)$, the velocity potential. The boundary conditions on u are:

$$\frac{\partial u}{\partial n} = \begin{cases} v, & \text{a constant, when } r < a,\ \theta = \pi/2, \\ 0, & r > a,\ \theta = \pi/2 \end{cases}$$

where $\partial/\partial n$ is the normal derivative, together with the radiation condition $\tau(iku + \partial u/\partial \tau) \to 0$ as $\tau \to \infty$. We now try to represent $u(r, \theta)$ as:

$$r^{1/2} u(r, \theta) = \int_L \mu g(\mu) P_{-1/2+\mu}(\cos\theta)\, J_\mu(kr)\, d\mu, \tag{2}$$

where $P_{1/2+\mu}(\cos\theta)$ is the Legendre function, $J_\mu(kr)$ the Bessel function, and $g(\mu)$ an unknown function of the complex variable $\mu = \sigma + i\tau$. L is a contour in a strip of finite width surrounding the imaginary μ axis from $\sigma - i\infty$ to $\sigma + i\infty$. The function $g(\mu)$ is to be determined by applying the boundary conditions to (2) and using the theorem of Kontorovich and Lebedev [3] which states that if

$$\varphi(kr) = \int_L \mu \Lambda(\mu) e^{i\pi\mu/2} J_\mu(kr)\, d\mu, \tag{3a}$$

then

$$\pi i \Lambda(\mu) = \int_0^\infty \varphi(kr) e^{-i\pi\mu/2} H_\mu^{(2)}(kr)\, dr/r. \tag{3b}$$

The conditions for validity of this theorem and other details can be found in the reference given.

One now applies the boundary conditions to (2) only to find that the resulting $\Lambda(\mu)$ is such that the conditions of the theorem are not satisfied and the integrals diverge However, there is an analogous theorem [6] in terms of modified Bessel functions of real argument which imposes considerably milder restrictions on $\Lambda(\mu)$. This suggests making the transition $k = -i\gamma$, γ real and positive, constructing a solution using the modified functions $I_\mu(\gamma r)$ and $K_\mu(\gamma r)$ and then returning to real k and the original Bessel functions. Obviously the integrals would still diverge if the contour L remains unchanged, but they will converge if L is first deformed to surround the positive real μ axis. This was recently demonstrated by Oberhettinger [4] and verified again by the result of the present problem.

The representation of $\mu(r, \theta)$ corresponding to real γ is

$$r^{1/2} u(r, \theta) = \int_L \mu g(\mu) P_{-1/2+\mu}(\cos\theta)\, I_\mu(\gamma r)\, d\mu. \tag{4}$$

The "baffle problem" can now be solved by means of the following Lebedev [6] theorem: If

$$\varphi(\gamma r) = \int_L \mu\Lambda(\mu)I_\mu(\gamma r)\,d\mu, \tag{5a}$$

then

$$\pi i\Lambda(\mu) = \int_0^\infty \varphi(\gamma r)K_\mu(\gamma r)\,dr/r. \tag{5b}$$

The conditions for validity are that $\Lambda(\mu)$ be an even function of μ, analytic in a strip of finite width including the imaginary axis and of decay at least as fast as

$$|\tau|^{-\frac{1}{2}-\epsilon}\exp(-\pi|\tau|/2), \quad \text{where} \quad \epsilon > 0.$$

2. Some properties of $I_\mu(\gamma r)$ and $K_\mu(\gamma r)$ as functions of μ. The functions $I_\mu(\gamma r)$ and $K_\mu(\gamma r)$ with real argument γr are entire functions with an infinite number of simple zeros [7]. For $|\mu/\gamma r| \gg 1$,

$$I_\mu(\gamma r) = [(\gamma r/2)^\mu/\Gamma(1+\mu)][1 + O(\mu^{-1})]. \tag{6}$$

Hence $I_\mu(\gamma r)$ decays rapidly as Re $\mu \to +\infty$. On the left half plane, as Re $\mu \to -\infty$, it has Γ function-like growth since $1/\Gamma(1+\mu) = -\sin\pi\mu\Gamma(-\mu)/\pi$. Along the imaginary axis $\mu = i\tau$, $I_\mu(\gamma r)$ has exponential growth as $\tau \to \pm\infty$.

The function $K_\mu(\gamma r)$ is defined by

$$K_\mu(\gamma r) = (\pi/2)[I_{-\mu}(\gamma r) - I_\mu(\gamma r)]/\sin\pi\mu. \tag{7}$$

As Re $\mu \to \infty$, $I_{-\mu}(\gamma r)$ is dominant and as Re $\mu \to -\infty$, $I_\mu(\gamma r)$ is dominant. Hence $K_\mu(\gamma r)$ has Γ function-like growth on both right and left half planes. Along the imaginary axis $K_\mu(\gamma r)$ decays exponentially as $\tau \to \pm\infty$. Asymptotic forms for $I_\mu(\gamma r)$ and hence for $K_\mu(\gamma r)$ can, of course, be found from (6) using Stirling's formula for $\Gamma(1+\mu)$.

3. Solution of the "baffle" problem. We represent $u(r,\theta)$ by means of (4) and enforce the boundary conditions as modified by transition to real γ:

$$\left.\begin{array}{ll} r \le a, & r^{3/2}v \\ r \ge a, & 0 \end{array}\right\} = \int_L \mu\Lambda(\mu)I_\mu(\gamma r)\,d\mu. \tag{8}$$

Here $\Lambda(\mu) = g(\mu)P'_{-\frac{1}{2}+\mu}(0)$, where the prime indicates differentiation with respect to the argument $\cos\theta$. The formal solution is given by the inversion integral (5b)·

$$\pi i\Lambda(\mu) = v\int_0^a r^{\frac{1}{2}}K_\mu(\gamma r)\,dr. \tag{9}$$

The integral defines an even function of μ and converges if μ is restricted to a strip of half-width 3/2 about the imaginary axis. Within this strip $\Lambda(\mu)$ is analytic and $|\Lambda(\mu)| \approx e^{-\pi|\tau|/2}/\tau^{3/2}$ as $|\tau| \to \infty$. From this it is seen that $\Lambda(\mu)$ satisfies the conditions of the Levedev theorem.

The integral (9) can be expressed in terms of the Lommel functions and they in terms of their series representation [8]. Thus one can continue $\Lambda(\mu)$ into the remainder of the μ plane where it is analytic except at the points $\mu = \pm(2n + 3/2)$, $n = 0, 1, 2, \cdots$, where it has simple poles with residues $[2^{1/2}\,v(-)^n i/\pi\gamma^{3/2}]\,[\Gamma(n + {}^{3/2})/n!]$. For Re $\mu \to \pm\infty$, we find also that $\Lambda(\mu) \approx I_{\mp\mu}(\gamma a)/(3/2 \pm \mu)\sin\pi\mu$. We now have $\Lambda(\mu)$ defined for the entire μ plane and are ready to convert the integral (4) into an eigenfunction expansion.

Substituting for $g(\mu)$ in (4) gives

$$r^{\frac{1}{2}}u(r,\theta) = \int_L \mu\Lambda(\mu)\frac{P_{-\frac{1}{2}+\mu}(\cos\theta)}{P_{-\frac{1}{2}+\mu}(0)} I_\mu(\gamma r)\,d\mu, \tag{10}$$

in which $\mu\Lambda(\mu)\ I_\mu(\gamma r) \approx (r/a)^\mu/\mu$ for large μ on the right half plane. Further the ratio of the Legendre functions behaves like $e^{(\theta-\pi/2)\mu}$ when $\theta \leq \pi/2$ the only values of θ in our problem. Hence for $r \leq a$ the contour Γ may be closed on the right and the integral is unchanged in value. The poles of the integrand are at $2n + 1/2$ and $2n + 3/2$ and the series of residues is the eigenfunction representation for the region $r \leq a$.

For $r \geq a$ the contour of (10) in its present form can not be closed. However the decomposition

$$\Lambda(\mu) = (\pi/2)[\lambda(-\mu) - \lambda(\mu)]/\sin\pi\mu$$

allows the contour to be closed. Then

$$\pi i\lambda(\mu) = v\int_0^a r^{\frac{1}{2}} I_\mu(\gamma r)\,dr, \tag{11}$$

and, by arguments similar to those used above for $\Lambda(\mu)$ we find $\lambda(\mu)$ to have poles at $-(2n + 3/2)$, analytic in the strip with growth like $I_\mu(\gamma a)/(3/2 - \mu)$, for large μ.

We now have

$$r^{\frac{1}{2}}u(r,\theta) = \int_L \mu\lambda(\mu)\frac{P_{-\frac{1}{2}+\mu}(\cos\theta)}{P'_{-\frac{1}{2}+\mu}(0)} K_\mu(\gamma r)\,d\mu, \tag{12}$$

where $\mu\lambda(\mu)\ K_\mu(\gamma r) \approx \mu^{-1}\ (\nu/a)^{-\mu}$, $|\mu| \to \infty$. Hence the contour can be closed on the right for $r \geq a$. The poles of the integrand lie at $2n + 1/2$, whose residues lead to the appropriate series of eigenfunctions in the space $r \geq a$. One sees now how the transition from integral representation to series of appropriate eigenfunctions resolves itself. When the transition from real γ to real k is made we see that the expansion will be in terms of Bessel functions for $r \leq a$, in terms of Hankel functions for $r \geq a$.

For completeness we record the eigenfunction expansion:

$$r \leq a:\quad u(r,\theta) = \sum_{n=0}^{\infty} a_n P_{2n+1}(\cos\theta) j_{2n+1}(kr) + \sum_{n=0}^{\infty} b_n P_{2n}(\cos\theta) j_{2n}(kr),$$

$$r \geq a:\quad u(r,\theta) = \sum_{n=0}^{\infty} c_n P_{2n}(\cos\theta)\, h_{2n}^{(2)}(kr),$$

where j and h stand for the spherical Bessel and Hankel functions and

$$a_n = (2v/k)(2n + 3/2)(-1)^n,$$

$$b_n = (2)^{\frac{1}{2}} vai(2n + \tfrac{1}{2})(-1)^{n+1}[\Gamma(n + \tfrac{1}{2})/n!]W\{H^{(2)}_{2n+\frac{1}{2}}(ka), s_{\frac{1}{2},2n+\frac{1}{2}}(ka)\},$$

$$c_n = ((2)^{\frac{1}{2}} iv/k)(2n + \tfrac{1}{2})(-1)^n[\Gamma(n + \tfrac{1}{2})/n!]\int_0^{ka} x^{\frac{1}{2}} J_{2n+\frac{1}{2}}(x)\,dx,$$

where W stands for Wronskian and $s_{\frac{1}{2},2n+\frac{1}{2}}$ is a Lommel function [8]. Note that for $r \geq a$ only the Legendre polynomials even in $\cos\theta$ appear, as required by the boundary condition.

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GEOMETRIC INTERPRETATION FOR THE RECIPROCAL DEFORMATION TENSORS*

By C. TRUESDELL (*Indiana University and Università di Bologna*)

In a finite deformation $\mathbf{x} = \mathbf{x}(\mathbf{X})$, changes of infinitesimal lengths may be measured by the tensor **C**, where

$$ds^2 = g_{km}\, dx^k\, dx^m = C_{KM}\, dX^K\, dX^M, \qquad C_{KM} = g_{km} x^k_{,K} x^m_{,M}\ , \tag{1}$$

or by the dual tensor **c** satisfying formulae that follow by systematic interchange of majuscules and minuscules. Geometric interpretations of **C** and **c** have been given by Cauchy and others. In 1894 Finger introduced the reciprocal tensors $\mathbf{C}^{-1}$ and $\mathbf{c}^{-1}$, and recent exact work on isotropic elastic bodies employs them often. While formulae such as

$$(C^{-1})^{KM} = g^{km} X^K_{,k} X^M_{,m} \tag{2}$$

for their expression and use are known, geometric interpretation has been lacking.

As is known, the correspondence between elements of area is given by $da^{km} = x^k_{,K}\, x^m_{,M}\, dA^{KM}$, where dA^{KM} is connected with the usual vector element of area dA_K by $dA_K = e_{KMP} dA^{MP}$, $e_{KMP} \equiv (\det\ G_{QR})^{1/2}\ \epsilon_{KMP}$. Hence[1]

$$\begin{aligned}
(da)^2 &= e^k_{pq} e_{krs} x^p_{,P} x^q_{,Q} x^r_{,R} x^s_{,S}\, dA^{PQ}\, dA^{RS}, \\
&= \frac{\det g_{uv}}{\det G_{UV}}\, g^{km}\left(\tfrac{1}{2}\epsilon_{krs}\epsilon^{KRS} x^r_{,R} x^s_{,S}\right)\left(\tfrac{1}{2}\epsilon_{mpq}\epsilon^{MPQ} x^p_{,P} x^q_{,Q}\right) dA_K\, dA_M\ , \\
&= \frac{\det g_{uv}}{\det G_{UV}} \left[\frac{\partial(x^1, x^2, x^3)}{\partial(X^1, X^2, X^3)}\right]^2 g^{km} X^K_{,k} X^M_{,m}\, dA_K\, dA_M\ , \\
&= [\det\ (C^{-1})^P_Q]^{-1} (C^{-1})^{KM}\, dA_K\, dA_M\ .
\end{aligned} \tag{3}$$

Comparing this result with (1) shows that the tensor $\mathbf{C}^{-1}/\det\ \mathbf{C}^{-1}$ measures changes of the magnitudes of infinitesimal areas in precisely the same way as **C** measures changes of infinitesimal lengths.

A known principle of duality, which may be called the *first* principle of duality,

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[1]A formula which is essentially the next to last step in (3) was given by Tonolo, Rend. sem. mat. Padova **14**, 43-117 (1943), Sec. V. **4**, but he did not mention any connection with $\mathbf{C}^{-1}$.

enables us to interchange the roles of **X** and **x** and thus obtain an analogous interpretation for $\mathbf{c}^{-1}/\det \mathbf{c}^{-1}$.

These results are equivalent to a *second principle of duality*: Any proposition on changes of length expressed in terms of **C** and **c** yields a theorem on changes of area if **C**, **c**, and "length" be replaced by $\mathbf{C}^{-1}/\det \mathbf{C}^{-1}$, $\mathbf{c}^{-1}/\det \mathbf{c}^{-1}$, and "area", respectively.

Of the many theorems that may be derived in this way, I record only one: The elements of area suffering extremal changes are normal to the principal directions of strain, and the greatest (least) change of area occurs in the plane normal to the axis of least (greatest) stretch; in fact, if the principal stretches dx/dX satisfy $\lambda_1 \geqq \lambda_2 \geqq \lambda_3$ the corresponding ratios da/dA satisfy $\lambda_2\lambda_3 \leqq \lambda_3\lambda_1 \leqq \lambda_1\lambda_2$. While this theorem is geometrically plausible, the first part does not seem obvious.

BOOK REVIEWS

(Continued from p. 394)

The theory of linear antennas. By Ronald W. P. King. Harvard University Press, Cambridge, 1956. xxi + 944 pp. $20.00.

The basic material in this treatise on the linear antenna is founded on a graduate course given by the author, and also includes both the theoretical and experimental data of many other workers in the antenna field. Although much of the information has been published previously in various technical journals, the detailed and extensive compilation and evaluation of so much of this work may be regarded as a worthwhile contribution to the antenna specialist. It is the belief of this reviewer, however, that the general usefulness might have been greatly enhanced if the material had been separated into two or more books.

The stated purpose of this book, which is basically mathematical in its approach, is to provide a bridge from the mathematician to the practical antenna engineer. An introduction summarizing the highlights in the historical development of the linear antenna theory is followed by a short chapter on the essentials of electromagnetic theory.

The mathematical difficulties in treating the behavior of a single linear radiator as end-load for a two-wire line are considered in chapter II. Only antennas having a cross section which is small compared with the wavelength are investigated in this text. Particular attention is paid to the end effect complications and cross-coupling between the line and the antenna. An approximate method of compensating for these effects is developed which employs appropriate lumped reactive elements at the junction between the line and the antenna. The characteristics of the isolated antenna are then studied in detail using several different formulations of the problem. The antenna impedance and admittance calculations are presented in a number of different types of graphical plots as well as in useful tables of numerical values. Unfortunately, in this section of particular interest to the practical engineer, a number of typographical errors occur in identifying the graphs.

Comparisons are made between theoretical calculations and experimental measurements obtained from various sources, and the constructional difficulties involved in precise measuring systems are discussed in considerable detail. In one reference in which this reviewer participated, however, it is noted that the author was incorrect in his statements describing the procedure.

Chapter III is devoted to a general investigation of the mutual coupling between antennas in various geometric configurations. An analysis is made of some of the more common types of antennas such as coplanar arrays, parasitic elements, folded dipoles, V-antennas, asymmetrically driven antennas, etc.

Chapter IV is devoted to the general analysis of the essential properties of receiving and scattering antennas. The freespace patterns and gains of various types of linear radiators are taken up in chapter V. Chapter VI discusses the electromagnetic fields of various configurations of linear radiators in commonly-used arrays.

Chapter VII is devoted to a study of the primary electromagnetic field and radiation characteristics

of antennas over a conducting earth. This portion could well have been omitted. The effect of the ionosphere is not taken up in this book.

In the last chapter, the antenna is formulated as a boundary-value problem with primary emphasis on the mathematical rather than the engineering approach. The radiation characteristics of hemispheroidal, conical, and cylindrical antennas are considered in detail. Recent works on large-angle conical antennas are not described in this book.

The appendix contains tables of generalized sine and cosine integrals useful in the theoretical calculations, a considerable number of problems based on the presented material, and a bibliography on antenna theory and measurements.

In general, the author seems inclined to favor the more complex approaches to the problems. For example, a simpler method of analysis for the ground plane antenna described in the literature* has not been included. Taken as a whole, however, the book provides a good background and reference for both the mathematician and applied scientist interested in this particular field. The book will be of relatively less usefulness to the general antenna engineer working with more complicated feed systems and antennas of large cross section which are beyond the scope of the simplified arrangements described.

O. M. Woodward, Jr.

Hydrodynamics. By Hugh L. Dryden, Francis D. Murnaghan and H. Bateman. Dover Publications, Inc., New York, 1956. 634 pp. $2.50 (paper-covered).

As is indicated on the reverse of the title page, this is a republication of Bulletin No. 84 of the National Research Council, which has long been out of print.

Deformation and flow of solids. Edited by R. Grammel. Springer-Verlag, Berlin, Gottingen, Heidelberg, 1956.

Work on plasticity of solids has developed in the past mainly along two lines:

1. Physical metallurgists and solid state physicists have attempted to interpret the basic plastic properties of materials in terms of their atomistic structure; the creation of the dislocation theory represents the most impressive example of this type of activity.

2. Mechanical engineers and applied mathematicians have attempted to establish certain general principles of a macroscopic nature for the deformation of solids; these principles permit, for example, computations of the plastic behavior of materials under more complicated cases of loading, if the behavior under simple loading conditions is known.

In both of these two branches of the theory of plasticity, much progress has been realized in the past ten years. However, there has been only little interaction between the two. Since it is felt by many that a close connection between the two branches would be fruitful for the development of either of them, the International Union of Theoretical and Applied Mechanics held a meeting in Madrid in 1955 on the deformation of solids, with the express purpose of including all aspects of the subject.

The book contains the lectures given at this meeting and the contributions to the discussion. The main topics are: theory of dislocations, theory of fracture, mathematical theory of plasticity, non-linear elasticity, viscoelasticity and relaxation. Some of the articles are in the form of surveys and others in the form of representative original papers. In this way the book easily provides a certain understanding of the present status of even those parts of the theory of plasticity with which the reader may be less familiar. It therefore should be of value to anyone who is interested in the deformation of solids.

K. Lücke

Handbuch Der Laplace-Transformation. By Gustav Doetsch. Band III. Birkhauser Verlag, Basel and Stuttgart, 1956. 300 pp. $9.35.

Volume 2 of this multi-volume work concluded with the treatment of ordinary differential equations and this third volume begins, appropriately, with a treatment of partial differential equations. After

*"An Ultra-High-Frequency Antenna of Simple Construction," G. H. Brown and J. Epstein, Communications, July 1940, p. 3.

a brief chapter defining the boundary value and initial value problems, a detailed treatment of the application of the Laplace transform to second order hyperbolic, parabolic, and elliptic equations with constant coefficients is given.

Briefly considered in the third chapter are the operational treatment of a variable coefficient equation in which (1) coefficients are independent of the transformation variable, and alternatively (2) the coefficients may be linear in the transformation variable. There follow chapters dealing with compatibility and uniqueness and with Huygens' and Eulers' principles.

Subsequent parts of the book are concerned with the application of the transform technique to difference equations and to integral equations whose kernel is necessarily of the convolution type, both for the finite and infinite domains.

The last chapter deals with the finite Laplace transform. Finally, an historical survey and a detailed bibliography are provided. The book should be of considerable interest and value to those who want a careful detailed spelling out of the use of the Laplace transform in connection with the problems mentioned above.

George F. Carrier

Theoretische Hydromechanik. By N. J. Kotschin, I. A. Kibel, and N. W. Rose. Translated from the Russian into German by K. Krienes. Akademie-Verlag, Berlin. Volume I, 1954, XII + 508 pp.; Volume II, 1955, VIII + 569 pp. $11.50 each volume.

The translation is of 1948 Russian edition. Volume I is concerned with the classical theory of the motion of an ideal fluid. The contents are as follows, the figures in brackets indicating the number of pages devoted to each topic; Chapter I, Kinematics of fluid motion: Deformation of a fluid drop (7): equation of continuity (15): kinematic character of irrotational and rotational flow (12). Chapter II, Basic equations of the hydrodynamics of an ideal fluid (36). Chapter III, Hydrostatics: Hydrostatic pressure (12): equilibrium of floating bodies (13). Chapter IV, the simplest motions of an ideal fluid: The integrals of Bernoulli and Cauchy (19): plane irrotational flow (14). Chapter, V, Vortex motion of an ideal fluid: The basic equations of vortex theory and the Helmholtz vorticity conservation theorems (27): the determination of the velocity field from a given vortex and source field (31): the Kármán vortex street (30). Chapter VI, The plane problem for the motion of a body in an ideal fluid (107). Chapter VII, The three-dimensional problem for the motion of a body in an ideal fluid (41). Chapter VIII, Wave motions of an ideal fluid: Basic equations of wave theory (7): plane waves (67): three-dimensional waves (21): long waves (31). The apparently disproportionate length of Chapter VI is explained by the fact that it treats also of free streamlines, thin aerofoils, and (in some detail) the planing of a plate at the surface of water. The plate here considered is of infinite breadth, although the case of the plate of finite breadth was solved by A. E. Green as long ago as 1936.

The subject is treated in great detail and there are some exercises which enhance the value of the work. On the other hand nothing like full use is made of vectors and the complex variable. Thus, for example, three pages are used to prove Kelvin's theorem on the constancy of circulation and Cartesian presentation is used throughout the section on plane waves. It is, however, gratifying to see convincing expositions of some of the physical deductions concerning such diverse matters as generation of vorticity, trade winds, breezes, and ocean currents due to variable salinity, which flow from the theorem of Bjerknes. This theorem expresses the rate of change of circulation in a circuit in terms of the number of unit tubes defined by isobars and isobulks which thread the circuit.

The contents of Volume II are as follows; Chapter I, Theoretical foundations of gasdynamics (by Kibel): The equations of gasdynamics (21): steady motion, the plane problem (143): steady motion, three-dimensional, including conical, flow (52): time dependent flow (24). Chapter II, The motion of viscous fluids (by Kotschin): Basic equations of motion of viscous fluids (46): exact solutions of the equations of motion (61): approximate solutions for small Reynolds number (42): approximate solutions for large Reynolds number, including boundary layer theory (86) and Oseen's theory of vanishing viscosity (24). Chapter III, Elements of turbulence theory (by Kibel): Turbulence and instability (27): fully developed turbulence (17): mean values of the hydrodynamic quantities (9).

In this volume also the treatment is detailed in some cases to the point of prolixity. Thus in Chapter II a good tensor notation could have reduced to about one the eleven pages which end up with the assumption that in a viscous fluid $3\lambda + 2\mu = 0$. It should be noted, and this applies to both volumes,

that 8 years have elapsed since the publication of the work in Russian so that it may be inferred that the treatise describes the state of the subject as it existed at least 10 years ago. In the case of Volume II therefore the reader must not be disappointed to miss the great advances which have been made in recent years. With this cautionary remark, it may be stated that the book gives a good and readable account of the classical parts of the subject matter listed above.

L. M. MILNE-THOMSON

Spheroidal wave functions. By J. A. Stratton, P. M. Morse, L. J. Chu, J. D. C. Little, and F. J. Corbato. The Technology Press of M.I.T., Cambridge, John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1956. xiii + 613 pp. $12.50.

In the foreword to these tables of coefficients of expansions of spheroidal wave functions, P. M. Morse points out that while it is known that there are eleven systems of coordinates which permit separation of the scalar wave equation, only three can be conveniently used in practice since only in the cases of rectangular, circular cylindrical, and spherical coordinates have the corresponding functions been tabulated. The solutions of the wave equation for elliptic cylindrical coordinates lead to Mathieu functions which have been treated in some detail in the literature, and for which some tables are available. The book under review presents corresponding material for spheroidal wave functions.

A section by Chu and Stratton discusses various expressions for these functions and mathematical relations between them. This is a reproduction of an earlier paper from the Journal of Mathematics and Physics (*XX*, 1941). The application of this material to the determination of the coefficients tabulated is presented in a second section by Little and Corbato. They discuss the numerical methods utilized, and emphasize the influence of the application of a high speed digital computer in carrying out this work. The computer was programmed and the output arranged to print the material in the form in which it appears in the book. This system avoids errors of computation and errors due to transcriptions between the calculated result and the printed tables.

The tables give the coefficients of the expansions of both oblate and prolate spheroidal radial functions in series of spherical Bessel functions, and of the corresponding angular functions in series of associated Legendre functions. References to auxiliary tables of spherical Bessel and associated Legendre functions are given. Tables of the spheroidal wave functions themselves were not presented at this time due to the volume of material involved with the many subsidiary parameters. The coefficients tabulated permit numerical values of the functions to be obtained quite quickly.

E. H. LEE

Engineering analysis. A survey of numerical procedures. By Stephen H. Crandall. McGraw-Hill Book Co., Inc., New York, Toronto, London, 1956. x + 417 pp. $9.50.

This well-written volume, latest member of the Engineering Societies Monographs series, is concerned with the analysis of complex engineering problems and the methods appropriate to their numerical solution. It is addressed to engineers or engineering mathematicians at the graduate level, and is based on lecture courses given by the author at the Imperial College of Science and Technology and the Massachusetts Institute of Technology. By far the largest part of the book is devoted to a survey of important numerical techniques; however, the prior task of reducing a problem to mathematical form is also recognized and discussed.

Perhaps the outstanding feature of the book is its organization, not according to a classification of numerical methods, but according to a classification of the problems to which they may be applied. Thus a given computational scheme, for example, relaxation, may be treated in several places. Three classes of problems are considered: equilibrium, eigenvalue, and propagation problems. The author devotes two chapters to each class, one dealing with systems with a finite number of degrees of freedom, the other with continuous systems. Each chapter contains roughly the following: (i) a derivation of several typical problems; (ii) a review of the salient mathematical characteristics of the type of problem

in question; (iii) a discussion of the most suitable techniques for obtaining numerical solutions, usually accompanied by examples worked out wholly or in part. Remarks concerning the convenience of particular methods for hand or machine computation are included, as are brief discussions of the possible errors involved.

In the opinion of the reviewer this book is a valuable contribution to the literature, particularly since it is written from the point of view of the engineer or applied mathematician who seeks to make use of numerical computations, rather than from that of the numerical analyst, who may be chiefly concerned with the refinement of the methods themselves.

Since this work is designed as a survey and not as an exhaustive treatise, the treatment may at times be somewhat condensed for the student entirely unfamiliar with the subject. Copious references to the existing literature, however, will guide the serious reader to more detailed accounts of those topics about which he requires more information. The book will be valuable as a reference to anyone concerned with the more practical aspects of numerical analysis, and should also be quite suitable as a text for a senior or graduate course in the field. Its worth in the latter connection is enhanced by the unusually large collection of exercises, some of which are drill problems, while others extend the ideas of the text. Most are accompanied by answers or hints as to their solution. The subject matter is well-indexed, and references to other works are provided by numerous footnotes, a selective bibliography, and an index of the authors cited. The physical appearance of the book is first rate, adhering to the standard set by previous members of the series. There are a number of minor errors, but the author has prepared a list of those which have come to his attention.

WILLIAM E. BOYCE

Symposium on Monte Carlo methods. Edited by Herbert A. Meyer. Held at the University of Florida. Conducted by the Statistical Laboratory. Sponsored by Wright Air Development Center of the Air Research and Development Command. March 16 and 17, 1954. John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1956. xvi + 382 pp. $7.50.

The proceedings of the symposium contains twenty papers covering a fairly wide range of topics including the generation of random numbers, general theory, and applications of Monte Carlo methods. Several papers are reviews written in a simple language for the novice whereas other are more specialized. Unfortunately, the papers are printed in the order of presentation at the symposium rather than in a logical order based upon content or difficulty. A reader unfamiliar with the subject could, however, understand nearly all the papers if he were to read them in a certain order starting with the reviews.

The book also contains an excellent bibliography divided into three sections. Part I lists paper and abstracts on "Monte Carlo Proper" including many rather inaccessible reports published by various laboratories. Part II lists references and abstracts on random digits and is meant to be fairly complete. Part III contains a selected list of articles and books (with some abstracts) mostly on related topics such as sampling.

Despite the fact that there is no obvious attempt to organize the material as would be expected of a text book, the end result is actually better than many books supposedly written in an organized way. It is certainly a valuable and timely contribution to the literature.

G. F. NEWELL

Einführung in die mathematische Statistik. By Dr. Leopold Schmetterer. Springer-Verlag, Wien, 1956. vi + 405 pp. $11.65.

The author, noting the lack of any modern book on mathematical statistics in the German language, presents a fairly complete survey of developments of the last quarter century. Although labeled an introduction, it is not an elementary level book. The reader is assumed to be familiar with matrix theory, calculus and elementary set theory. With this as a starting point, the author concentrates on the subject at hand with a minimum of distraction.

About a third of the book is an introduction to probability theory, primarily an outline of the modern

approach with many theorems stated without proof. This is followed by chapters on elementary sample theory, confidence regions, theory of parameter estimating, introduction to test theory, regression theory, introduction to nonparametric theory, and the classical Bayes method. There is also a German to English translation of most of the technical words.

The style of writing is concise, easy to translate, and orderly. Although certain important topics may be left out, no topic is treated at excessive length. The book is well suited as a text book even for English speaking students preferably at the graduate level in American universities.

G. Newell

Elasticity, fracture and flow, with engineering and geological applications. By J. C. Jaeger. Methuen & Co., Ltd., London; John Wiley & Sons, New York, 1956. viii + 152 pp. $2.50.

A thorough study of the mathematical theory of elasticity is often held to be the obvious first step in exploring the mechanics of deformable solids. In favor of this view speaks the fact that the mathematical theory of elasticity sets standards of mathematical rigor that have not yet been, and probably never will be, attained in other branches of solid mechanics. On the other hand, prolonged exclusive exposure to the theory of elasticity may foster patterns of thinking that will make it unnecessarily difficult for the student to absorb the differing viewpoints of other branches of mechanics of solids. In recognition of this danger, several schools are currently experimenting with the idea of a general introductory course in mechanics of continua that is to be followed by more specialized courses on the various branches of mechanics of solids and fluids. The lack of suitable textbooks has severely handicapped these efforts. The present volume fills this gap for solids.

The first chapter (48 pp.) is concerned with the analysis of stress, and finite and infinitesimal strain. In the second chapter (68 pp.), the author discusses the mechanical behavior of actual materials and introduces suitable mathematical models of elastic, viscous, plastic, and visco-plastic behavior. The third chapter (43 pp.) treats a number of typical elementary problems concerning equilibrium and motion of these solids.

The exposition is clear and easy to follow. Throughout the book, the emphasis is on the basic assumptions and the manner in which these affect the solutions.

W. Prager

Modern mathematics for the engineer. Edited by E. F. Beckenbach. McGraw-Hill Book Company, Inc., New York, 1956. xx + 514 pp. $7.50.

The chapters of this book correspond to a series of invitation lectures given at the University of California, Los Angeles. The book consists of three parts. Part I is entitled Mathematical Models and contains the following chapters: Linear and Nonlinear Oscillations (S. Lefschetz), Equilibrium Analysis: The Stability Theory of Poincaré and Liapunov (R. Bellman), Exterior Ballistics (J. W. Green), Elements of Calculus of Variations (M. R. Hestenes), Hyperbolic Partial Differential Equations and Applications (R. Courant), Boundary Value Problems in Elliptic Partial Differential Equations (M. M. Schiffer), The Elastostatic Boundary-Value Problems (I. S. Sokolnikoff). Part II is devoted to Probabilistic Problems; it contains the following chapters: The Theory of Prediction (N. Wiener), The Theory of Games (H. F. Bohnenblust), Applied Mathematics in Operations Research (G. W. King), The Theory of Dynamic Programming (R. Bellman), Monte Carlo Methods (G. W. Brown). Part III has the title Computational Considerations and consists of the following chapters: Matrices in Engineering (L. A. Pipes), Functional Transformations for Engineering Design (J. L. Barnes), Conformal Mapping Methods (E. F. Beckenbach), Nonlinear Methods (C. B. Morrey, Jr.), What Are Relaxation Methods? (G. E. Forsythe), Methods of Steepest Descent (C. B. Tompkins), High-Speed Computing Devices and Their Applications (D. H. Lehmer).

W. Prager

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